ON THE UNIQUENESS OF THE CANONICAL POLYADIC DECOMPOSITION OF THIRD-ORDER TENSORS — PART II: UNIQUENESS OF THE OVERALL DECOMPOSITION *

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Abstract. Canonical Polyadic (also known as Candecomp/Parafac) Decomposition (CPD) of a higher-order tensor is decomposition in a minimal number of rank-1 tensors. In Part I, we gave an overview of existing results concerning uniqueness and presented new, relaxed, conditions that guarantee uniqueness of one factor matrix. In Part II we use these results for establishing overall CPD uniqueness in cases where none of the factor matrices has full column rank. We obtain uniqueness conditions involving Khatri-Rao products of compound matrices and Kruskal-type conditions. We consider both deterministic and generic uniqueness. We also discuss uniqueness of INDSCAL and other constrained polyadic decompositions.

Key words. Canonical Polyadic Decomposition, Candecomp, Parafac, three-way array, tensor, multilinear algebra, Khatri-Rao product, compound matrix

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1. Introduction.

1.1. Problem statement. Throughout the paper $\mathbb{F}$ denotes the field of real or complex numbers; $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ denote conjugate, transpose, and conjugate transpose, respectively; $r_{\mathbf{A}}$, range$(\mathbf{A})$, and ker$(\mathbf{A})$ denote the rank, the range, and the null space of a matrix $\mathbf{A}$, respectively; Diag$(\mathbf{d})$ denotes a square diagonal matrix with the elements of a vector $\mathbf{d}$ on the main diagonal; span$\{\mathbf{f}_1,\dots,\mathbf{f}_k\}$ denotes the linear span of the vectors $\mathbf{f}_1,\dots,\mathbf{f}_k$; $\mathbf{e}_r^R$ denotes the r-th vector of the canonical basis of $\mathbb{F}^R$; C_n^k denotes the binomial coefficient, $C_n^k=\frac{n!}{k!(n-k)!}$; $\mathbf{O}_{m\times n}$, $\mathbf{0}_m$, and $\mathbf{I}_n$ are the zero $m\times n$ matrix, the zero $m\times 1$ vector, and the $n\times n$ identity matrix, respectively.

We have the following basic definitions. A third-order tensor $\mathcal{T}=(t_{ijk})\in\mathbb{F}^{I\times J\times K}$ is *rank*-1 if there exist three nonzero vectors $\mathbf{a}\in\mathbb{F}^I$, $\mathbf{b}\in\mathbb{F}^J$ and $\mathbf{c}\in\mathbb{F}^K$ such that $\mathcal{T}=\mathbf{a}\circ\mathbf{b}\circ\mathbf{c}$, in which "$\circ$" denotes the *outer product*. That is, $t_{ijk}=a_ib_jc_k$ for all values of the indices.

A *Polyadic Decomposition* (PD) of a third-order tensor $\mathcal{T}\in\mathbb{F}^{I\times J\times K}$ expresses $\mathcal{T}$ as a sum of rank-1 terms:

$$\mathcal{T}=\sum_{r=1}^{R}\mathbf{a}_r\circ\mathbf{b}_r\circ\mathbf{c}_r, \tag{1.1}$$

where $\mathbf{a}_r\in\mathbb{F}^I$, $\mathbf{b}_r\in\mathbb{F}^J$, $\mathbf{c}_r\in\mathbb{F}^K$ are nonzero vectors.

We call the matrices $\mathbf{A}=\begin{bmatrix}\mathbf{a}_1 & \dots & \mathbf{a}_R\end{bmatrix}\in\mathbb{F}^{I\times R}$, $\mathbf{B}=\begin{bmatrix}\mathbf{b}_1 & \dots & \mathbf{b}_R\end{bmatrix}\in\mathbb{F}^{J\times R}$ and $\mathbf{C}=\begin{bmatrix}\mathbf{c}_1 & \dots & \mathbf{c}_R\end{bmatrix}\in\mathbb{F}^{K\times R}$ the *first, second* and *third factor matrix* of $\mathcal{T}$,

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respectively. We also write (1.1) as $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$.

DEFINITION 1.1. *The* rank *of a tensor* $\mathcal{T} \in \mathbb{F}^{I\times J\times K}$ *is defined as the minimum number of rank-*1 *tensors in a PD of* $\mathcal{T}$ *and is denoted by* $r_{\mathcal{T}}$.

DEFINITION 1.2. *A* Canonical Polyadic Decomposition *(CPD) of a third-order tensor* $\mathcal{T}$ *expresses* $\mathcal{T}$ *as a minimal sum of rank-*1 *terms.*

Note that $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ is a CPD of $\mathcal{T}$ if and only if $R = r_{\mathcal{T}}$.

Let us reshape $\mathcal{T}$ into a matrix $\mathbf{T} \in \mathbb{F}^{IJ\times K}$ as follows: the (i, j, k)-th entry of $\mathcal{T}$ corresponds to the $((i-1)J + j, k)$-th entry of $\mathbf{T}$. In particular, the rank-1 tensor $\mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$ corresponds to the rank-1 matrix $(\mathbf{a} \otimes \mathbf{b})\mathbf{c}^T$, in which "$\otimes$" denotes the *Kronecker product*. Thus, (1.1) can be identified with

$$\mathbf{T}^{(1)} := \mathbf{T} = \sum_{r=1}^{R} (\mathbf{a}_r \otimes \mathbf{b}_r)\mathbf{c}_r^T = [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \cdots \ \mathbf{a}_R \otimes \mathbf{b}_R]\mathbf{C}^T = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T, \tag{1.2}$$

in which "$\odot$" denotes the *Khatri-Rao product* or column-wise Kronecker product. Similarly, one can reshape $\mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$ into any of the matrices

$$(\mathbf{b} \otimes \mathbf{c})\mathbf{a}^T, \quad (\mathbf{c} \otimes \mathbf{a})\mathbf{b}^T, \quad (\mathbf{a} \otimes \mathbf{c})\mathbf{b}^T, \quad (\mathbf{b} \otimes \mathbf{a})\mathbf{c}^T, \quad (\mathbf{c} \otimes \mathbf{b})\mathbf{a}^T$$

and obtain the factorizations

$$\mathbf{T}^{(2)} = (\mathbf{B} \odot \mathbf{C})\mathbf{A}^T, \qquad \mathbf{T}^{(3)} = (\mathbf{C} \odot \mathbf{A})\mathbf{B}^T, \qquad \mathbf{T}^{(4)} = (\mathbf{A} \odot \mathbf{C})\mathbf{B}^T \qquad \text{etc.} \tag{1.3}$$

The matrices $\mathbf{T}^{(1)}, \mathbf{T}^{(2)}, \dots$ are called *the matrix representations* or *matrix unfoldings* of the tensor $\mathcal{T}$.

It is clear that in (1.1)–(1.2) the rank-1 terms can be arbitrarily permuted and that vectors within the same rank-1 term can be arbitrarily scaled provided the overall rank-1 term remains the same. *The CPD of a tensor is unique* when it is only subject to these trivial indeterminacies. Formally, we have the following definition.

DEFINITION 1.3. *Let* $\mathcal{T}$ *be a tensor of rank* R. *The CPD of* $\mathcal{T}$ *is* essentially unique *if* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_R$ *implies that there exist an* $R \times R$ *permutation matrix* $\mathbf{\Pi}$ *and* $R \times R$ *nonsingular diagonal matrices* $\mathbf{\Lambda}_{\mathbf{A}}$, $\mathbf{\Lambda}_{\mathbf{B}}$, *and* $\mathbf{\Lambda}_{\mathbf{C}}$ *such that*

$$\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}}, \quad \bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{B}}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}, \quad \mathbf{\Lambda}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{B}}\mathbf{\Lambda}_{\mathbf{C}} = \mathbf{I}_R.$$

PDs can also be partially unique. That is, a factor matrix may be essentially unique without the overall PD being essentially unique. We will resort to the following definition.

DEFINITION 1.4. *Let* $\mathcal{T}$ *be a tensor of rank* R. *The first (resp. second or third) factor matrix of* $\mathcal{T}$ *is* essentially unique *if* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_R$ *implies that there exist an* $R \times R$ *permutation matrix* $\mathbf{\Pi}$ *and an* $R \times R$ *nonsingular diagonal matrix* $\mathbf{\Lambda}_{\mathbf{A}}$ *(resp.* $\mathbf{\Lambda}_{\mathbf{B}}$ *or* $\mathbf{\Lambda}_{\mathbf{C}}$*) such that*

$$\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}} \quad \textit{(resp.} \quad \bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{B}} \qquad \textit{or} \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}).$$

For brevity, in the sequel we drop the term "essential", both when it concerns the uniqueness of the overall CPD and when it concerns the uniqueness of one factor matrix.

In this paper we present both deterministic and generic uniqueness results. Deterministic conditions concern one particular PD $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. For generic uniqueness we resort to the following definitions.

DEFINITION 1.5. *Let* μ *be the Lebesgue measure on* $\mathbb{F}^{(I+J+K)R}$. *The CPD of an* $I \times J \times K$ *tensor of rank* R *is generically unique if*

$$\mu\{(\mathbf{A}, \mathbf{B}, \mathbf{C}) : \textit{ the CPD of the tensor } [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R \textit{ is not unique } \} = 0.$$

Definition 1.6. *Let μ be the Lebesgue measure on $\mathbb{F}^{(I+J+K)R}$. The first (resp. second or third) factor matrix of an $I \times J \times K$ tensor of rank R is generically unique if*

$$\mu\left\{(\mathbf{A},\mathbf{B},\mathbf{C}) : \begin{array}{l}\textit{the first (resp. second or third) factor matrix}\\ \textit{of the tensor } [\mathbf{A},\mathbf{B},\mathbf{C}]_R \textit{ is not unique}\end{array}\right\} = 0.$$

Let the matrices $\mathbf{A} \in \mathbb{F}^{I\times R}$, $\mathbf{B} \in \mathbb{F}^{J\times R}$ and $\mathbf{C} \in \mathbb{F}^{K\times R}$ be randomly sampled from a continuous distribution. Generic uniqueness then means uniqueness that holds with probability one.

1.2. Literature overview. We refer to the overview papers [3, 6, 12] and the references therein for background, applications and algorithms for CPD. Here, we focus on results concerning uniqueness of the CPD.

1.2.1. Deterministic conditions. We refer to [7, Subsection 1.2] for a detailed overview of deterministic conditions. Here we just recall three Kruskal theorems and new results from [7] that concern the uniqueness of one factor matrix. To present Kruskal's theorem we recall the definition of k-rank.

Definition 1.7. *The k-rank of a matrix $\mathbf{A}$ is the largest number $k_{\mathbf{A}}$ such that every subset of $k_{\mathbf{A}}$ columns of the matrix $\mathbf{A}$ is linearly independent.*

Kruskal's theorem states the following.

Theorem 1.8. *[14, Theorem 4a, p. 123] Let $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ and let*

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2R + 2. \tag{1.4}$$

Then $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ is unique.

Kruskal also obtained the following more general results which are less known.

Theorem 1.9. *[14, Theorem 4b, p. 123] (see also Corollary 1.29 below) Let $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ and let*

$$\begin{aligned}
\min(k_{\mathbf{A}}, k_{\mathbf{C}}) + r_{\mathbf{B}} &\geq R + 2,\\
\min(k_{\mathbf{A}}, k_{\mathbf{B}}) + r_{\mathbf{C}} &\geq R + 2,\\
r_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} &\geq 2R + 2 + \min(r_{\mathbf{A}} - k_{\mathbf{A}}, r_{\mathbf{B}} - k_{\mathbf{B}}),\\
r_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} &\geq 2R + 2 + \min(r_{\mathbf{A}} - k_{\mathbf{A}}, r_{\mathbf{C}} - k_{\mathbf{C}}).
\end{aligned}$$

Then $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ is unique.

Let the matrices $\mathbf{A}$ and $\mathbf{B}$ have R columns. Let $\tilde{\mathbf{A}}$ be any set of columns of $\mathbf{A}$, let $\tilde{\mathbf{B}}$ be the corresponding set of columns of $\mathbf{B}$, and define

$$H_{\mathbf{AB}}(\delta) := \min_{card(\tilde{\mathbf{A}})=\delta} [r_{\tilde{\mathbf{A}}} + r_{\tilde{\mathbf{B}}} - \delta] \quad \text{for} \quad \delta = 1, 2, \dots, R.$$

We will say that condition (Hm) holds for the matrices $\mathbf{A}$ and $\mathbf{B}$ if

$$H_{\mathbf{AB}}(\delta) \geq \min(\delta, m) \quad \text{for} \quad \delta = 1, 2, \dots, R. \tag{Hm}$$

The following Theorem is the strongest result about uniqueness from [14].

Theorem 1.10. *[14, Theorem 4e, p. 125](see also Corollary 1.27 below) Let $\mathcal{T} = [\mathbf{A},\mathbf{B},\mathbf{C}]_R$ and let $m_{\mathbf{B}} := R - r_{\mathbf{B}} + 2$, $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$. Assume that*

(i) (H1) *holds for $\mathbf{B}$ and $\mathbf{C}$;*

(ii) ($\mathrm{H}\mathrm{m}_{\mathbf{B}}$) *holds for* $\mathbf{C}$ *and* $\mathbf{A}$*;*
(iii) ($\mathrm{H}\mathrm{m}_{\mathbf{C}}$) *holds for* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Then $r_{\mathcal{T}} = R$ *and the CPD of* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *is unique.*

For the formulation of other results we recall the definition of compound matrix.

Definition 1.11. *[7, Definition 2.1 and Example 2.2] The k-th compound matrix of $I \times R$ matrix $\mathbf{A}$ (denoted by $\mathcal{C}_k(\mathbf{A})$) is the $C_I^k \times C_R^k$ matrix containing the determinants of all $k \times k$ submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order.*

With a vector $\mathbf{d} = \begin{bmatrix} d_1 & \dots & d_R \end{bmatrix}^T$ we associate the vector

$$\widehat{\mathbf{d}}^m := \begin{bmatrix} d_1 \cdots d_m & d_1 \cdots d_{m-1} d_{m+1} & \dots & d_{R-m+1} \cdots d_R \end{bmatrix}^T \in \mathbb{F}^{C_R^m}, \tag{1.5}$$

whose entries are all products $d_{i_1} \cdots d_{i_m}$ with $1 \leq i_1 < \cdots < i_m \leq R$. Let us define conditions (Km), (Cm), (Um) and (Wm), which depend on matrices $\mathbf{A} \in \mathbb{F}^{I \times R}$, $\mathbf{B} \in \mathbb{F}^{J \times R}$, $\mathbf{C} \in \mathbb{F}^{K \times R}$ and an integer parameter m:

$$\begin{cases} r_{\mathbf{A}} + k_{\mathbf{B}} & \geq R + m, \\ k_{\mathbf{A}} & \geq m \end{cases} \quad \text{or} \quad \begin{cases} r_{\mathbf{B}} + k_{\mathbf{A}} & \geq R + m, \\ k_{\mathbf{B}} & \geq m \end{cases}; \tag{Km}$$

$$\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B}) \quad \text{has full column rank;} \tag{Cm}$$

$$\begin{cases} (\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B}))\widehat{\mathbf{d}}^m = \mathbf{0}, \\ \mathbf{d} \in \mathbb{F}^R \end{cases} \Rightarrow \quad \widehat{\mathbf{d}}^m = \mathbf{0}; \tag{Um}$$

$$\begin{cases} (\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B}))\widehat{\mathbf{d}}^m = \mathbf{0}, \\ \mathbf{d} \in \operatorname{range}(\mathbf{C}^T) \end{cases} \Rightarrow \quad \widehat{\mathbf{d}}^m = \mathbf{0}. \tag{Wm}$$

In the sequel, we will for instance say that "condition (Um) holds for the matrices $\mathbf{X}$ and $\mathbf{Y}$" if condition (Um) holds for the matrices $\mathbf{A}$ and $\mathbf{B}$ replaced by the matrices $\mathbf{X}$ and $\mathbf{Y}$, respectively. We will simply write (Um) (resp. (Km),(Hm),(Cm) or (Wm)) when no confusion is possible.

It is known that conditions (K2), (C2), (U2) guarantee uniqueness of the CPD with full column rank in the third mode (see Proposition 1.15 below), and that condition (Km) guarantees the uniqueness of the third factor matrix [8], [7, Theorem 1.12].

In the following Proposition we gather, for later reference, properties of conditions (Km), (Cm), (Um) and (Wm) that were established in [7, §2–§3]. The proofs follow from properties of compound matrices [7, Subsection 2.1].

Proposition 1.12.

(1) If (Km) *holds, then* (Cm) *and* (Hm) *hold [7, Lemmas 3.8, 3.9];*
(2) if (Cm) *or* (Hm) *holds, then* (Um) *holds [7, Lemmas 3.1, 3.10];*
(3) if (Um) *holds, then* (Wm) *holds [7, Lemma 3.3];*
(4) if (Km) *holds, then* (Kk) *holds for* $k \leq m$ *[7, Lemma 3.4];*
(5) if (Hm) *holds, then* (Hk) *holds for* $k \leq m$ *[7, Lemma 3.5];*
(6) if (Cm) *holds, then* (Ck) *holds for* $k \leq m$ *[7, Lemma 3.6];*
(7) if (Um) *holds, then* (Uk) *holds for* $k \leq m$ *[7, Lemma 3.7];*
(8) if (Wm) *holds and* $\min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m - 1$*, then* (Wk) *holds for* $k \leq m$ *[7, Lemma 3.12];*
(9) if (Um) *holds, then* $\min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m$ *[7, Lemma 2.8].*

The following schemes illustrate Proposition 1.12:

$$
\begin{array}{cccccccccccc}
 & & (\mathrm{W_m}) & & (\mathrm{W_{m\text{-}1}}) & & \dots & & (\mathrm{W_2}) & & (\mathrm{W_1}) & \\
 & & \Uparrow & & \Uparrow & & \dots & & \Uparrow & & \Uparrow & \\
\begin{cases} k_{\mathbf{A}} \geq m, \\ k_{\mathbf{B}} \geq m \end{cases} & \Leftarrow & (\mathrm{U_m}) & \Rightarrow & (\mathrm{U_{m\text{-}1}}) & \Rightarrow & \dots & \Rightarrow & (\mathrm{U_2}) & \Rightarrow & (\mathrm{U_1}) & \\
 & & \Uparrow & & \Uparrow & & \dots & & \Uparrow & & \Updownarrow & , \\
 & & (\mathrm{C_m}) & \Rightarrow & (\mathrm{C_{m\text{-}1}}) & \Rightarrow & \dots & \Rightarrow & (\mathrm{C_2}) & \Rightarrow & (\mathrm{C_1}) & \\
 & & \Uparrow & & \Uparrow & & \dots & & \Uparrow & & \Uparrow & \\
 & & (\mathrm{K_m}) & \Rightarrow & (\mathrm{K_{m\text{-}1}}) & \Rightarrow & \dots & \Rightarrow & (\mathrm{K_2}) & \Rightarrow & (\mathrm{K_1}) &
\end{array}
\tag{1.6}
$$

and

$$
\text{if } \min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m-1, \text{ then } \quad (\mathrm{W_m}) \Rightarrow (\mathrm{W_{m\text{-}1}}) \Rightarrow \dots \Rightarrow (\mathrm{W_2}) \Rightarrow (\mathrm{W_1}). \tag{1.7}
$$

Scheme (1.6) also remains valid after replacing conditions $(\mathrm{C_m}),\dots,(\mathrm{C_1})$ and equivalence $(\mathrm{C_1}) \Leftrightarrow (\mathrm{U_1})$ by conditions $(\mathrm{H_m}),\dots,(\mathrm{H_1})$ and implication $(\mathrm{H_1}) \Rightarrow (\mathrm{U_1})$, respectively. One can easily construct examples where $(\mathrm{C_m})$ holds but $(\mathrm{H_m})$ does not hold. We do not know examples where $(\mathrm{H_m})$ is more relaxed than $(\mathrm{C_m})$.

Deterministic results concerning the uniqueness of one particular factor matrix were presented in [7, §4]. We first have the following proposition.

Proposition 1.13. *[7, Proposition 4.9] Let $\mathbf{A} \in \mathbb{F}^{I\times R}$, $\mathbf{B} \in \mathbb{F}^{J\times R}$, $\mathbf{C} \in \mathbb{F}^{K\times R}$, and let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. Assume that*

(i) $k_{\mathbf{C}} \geq 1$;

(ii) $m = R - r_{\mathbf{C}} + 2 \leq \min(I, J)$;

(iii) $\mathbf{A} \odot \mathbf{B}$ *has full column rank;*

(iv) *the triplet of matrices* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *satisfies conditions* $(\mathrm{W_m}), \dots, (\mathrm{W_1})$.

Then $r_{\mathcal{T}} = R$ and the third factor matrix of $\mathcal{T}$ is unique.

Combining Propositions 1.12 and 1.13 we obtained the following result.

Proposition 1.14. *[7, Proposition 4.3, Corollaries 4.4 and 4.5] Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathcal{T}$ be as in Proposition 1.13. Assume that $k_{\mathbf{C}} \geq 1$ and $m = m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$. Then*

$$
\begin{tikzcd}
 & & (\mathrm{C_m}) \arrow[dr, Rightarrow, "(1.6)"] & & \\
(1.4) \arrow[r, Rightarrow, "trivial"] & (\mathrm{K_m}) \arrow[ur, Rightarrow, "(1.6)"] \arrow[dr, Rightarrow, "(1.6)"'] & & (\mathrm{U_m}) \arrow[r, Rightarrow, "(1.6)"] & \begin{cases} (\mathrm{C_1}) \\ \min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m-1, \\ (\mathrm{W_m}) \end{cases} \\
 & & (\mathrm{H_m}) \arrow[ur, Rightarrow, "(1.6)"'] & &
\end{tikzcd}
\tag{1.8}
$$

$$
\xRightarrow{(1.7)} \begin{cases} (\mathrm{C_1}) \\ (\mathrm{W_1}), \dots, (\mathrm{W_m}) \end{cases} \Rightarrow \begin{cases} r_{\mathcal{T}} = R, \\ \textit{the third factor matrix of } \mathcal{T} \textit{ is unique.} \end{cases}
$$

Note that for $r_{\mathbf{C}} = R$, we have $m = 2$ and $(\mathrm{U_2})$ is equivalent to $(\mathrm{W_2})$. Moreover, in this case $(\mathrm{U_2})$ is necessary for uniqueness. We obtain the following counterpart of Proposition 1.14.

Proposition 1.15. *[4, 10, 15] Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathcal{T}$ be as in Proposition 1.13. Assume that $r_{\mathbf{C}} = R$. Then*

$$
\begin{tikzcd}
 & & (\mathrm{C_2}) \arrow[dr, Rightarrow] & & \\
(1.4) \arrow[r, Rightarrow] & (\mathrm{K_2}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (\mathrm{U_2}) \arrow[r, Leftrightarrow] & \begin{cases} r_{\mathcal{T}} = R, \\ \textit{the CPD of } \mathcal{T} \textit{ is unique.} \end{cases} \\
 & & (\mathrm{H_2}) \arrow[ur, Rightarrow] & &
\end{tikzcd}
\tag{1.9}
$$

1.2.2. Generic conditions. Let the matrices $\mathbf{A} \in \mathbb{F}^{I\times R}$, $\mathbf{B} \in \mathbb{F}^{J\times R}$ and $\mathbf{C} \in \mathbb{F}^{K\times R}$ be randomly sampled from a continuous distribution. It can be easily checked that the equations

$$k_{\mathbf{A}} = r_{\mathbf{A}} = \min(I, R), \quad k_{\mathbf{B}} = r_{\mathbf{B}} = \min(J, R), \quad k_{\mathbf{C}} = r_{\mathbf{C}} = \min(K, R)$$

hold generically. Thus, by (1.4), the CPD of an $I \times J \times K$ tensor of rank R is generically unique if

$$\min(I, R) + \min(J, R) + \min(K, R) \geq 2R + 2. \tag{1.10}$$

The generic uniqueness of one factor matrix has not yet been studied as such. It can be easily seen that in (1.8) the generic version of (Km) for $m = R - K + 2$ is also given by (1.10).

Let us additionally assume that $K \geq R$. Under this assumption, (1.10) reduces to

$$\min(I, R) + \min(J, R) \geq R + 2.$$

The generic version of condition (C2) was given in [4, 16]. It was indicated that the $C_I^2 C_J^2 \times C_R^2$ matrix $\mathbf{U} = \mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})$ generically has full column rank whenever the number of columns of $\mathbf{U}$ does not exceed the number of rows. By Proposition 1.15 the CPD of an $I \times J \times K$ tensor of rank R is then generically unique if

$$K \geq R \qquad \text{and} \quad I(I-1)J(J-1)/4 = C_I^2 C_J^2 \geq C_R^2 = R(R-1)/2. \tag{1.11}$$

The four following results have been obtained in algebraic geometry.

THEOREM 1.16. *[18, Corollary 3.7] Let $3 \leq I \leq J \leq K$, $K-1 \leq (I-1)(J-1)$, and let K be odd. Then the CPD of an $I \times J \times K$ tensor of rank R is generically unique if $R \leq IJK/(I+J+K-2) - K$.*

THEOREM 1.17. *[2, Theorem 1.1] Let $I \leq J \leq K$. Let α, β be maximal such that $2^\alpha \leq I$ and $2^\beta \leq J$. Then the CPD of an $I\times J\times K$ tensor of rank R is generically unique if $R \leq 2^{\alpha+\beta-2}$.*

THEOREM 1.18. *[2, Proposition 5.2], [18, Theorem 2.7] Let $R \leq (I-1)(J-1) \leq K$. Then the CPD of an $I \times J \times K$ tensor of rank R is generically unique.*

THEOREM 1.19. *[2, Theorem 1.2] The CPD of an $I \times I \times I$ tensor of rank R is generically unique if $R \leq k(I)$, where $k(I)$ is given in Table 1.1.*

TABLE 1.1
Upper bound $k(I)$ on R under which generic uniqueness of the CPD of a $I \times I \times I$ tensor is guaranteed by Theorem 1.19.

I	2	3	4	5	6	7	8	9	10
$k(I)$	2	3	5	9	13	18	22	27	32

Finally, for a number of specific cases of dimensions and rank, generic uniqueness results have been obtained in [19].

1.3. Results and organization. In this paper we use the conditions in (1.8) to establish CPD uniqueness in cases where $r_{\mathbf{C}} < R$.

In §2 we assume that a tensor admits two PDs that have one or two factor matrices in common. We establish conditions under which both decompositions are the same. We obtain the following results.

PROPOSITION 1.20. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}]_R$*, where* $\mathbf{\Pi}$ *is an* $R \times R$ *permutation matrix and* $\mathbf{\Lambda}_{\mathbf{C}}$ *is a nonsingular diagonal matrix. Let the matrices* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *satisfy the following condition*

$$\max(\min(k_{\mathbf{A}}, k_{\mathbf{B}} - 1),\ \min(k_{\mathbf{A}} - 1, k_{\mathbf{B}})) + k_{\mathbf{C}} \geq R + 1. \tag{1.12}$$

Then there exist nonsingular diagonal matrices $\mathbf{\Lambda}_{\mathbf{A}}$ *and* $\mathbf{\Lambda}_{\mathbf{B}}$ *such that*

$$\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}}, \qquad \bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{B}}, \qquad \mathbf{\Lambda}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{B}}\mathbf{\Lambda}_{\mathbf{C}} = \mathbf{I}_R.$$

PROPOSITION 1.21. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\mathbf{A}\mathbf{\Pi}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{A}}, \bar{\mathbf{B}}, \mathbf{C}\mathbf{\Pi}_{\mathbf{C}}\mathbf{\Lambda}_{\mathbf{C}}]_R$*, where* $\mathbf{\Pi}_{\mathbf{A}}$ *and* $\mathbf{\Pi}_{\mathbf{C}}$ *are* $R \times R$ *permutation matrices and where* $\mathbf{\Lambda}_{\mathbf{A}}$ *and* $\mathbf{\Lambda}_{\mathbf{C}}$ *are nonsingular diagonal matrices. Let the matrices* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *satisfy at least one of the following conditions*

$$\begin{aligned} &k_{\mathbf{C}} \geq 2 \quad \textit{and} \quad \max(\min(k_{\mathbf{A}}, k_{\mathbf{B}} - 1),\ \min(k_{\mathbf{A}} - 1, k_{\mathbf{B}})) + r_{\mathbf{C}} \geq R + 1, \\ &k_{\mathbf{A}} \geq 2 \quad \textit{and} \quad \max(\min(k_{\mathbf{B}}, k_{\mathbf{C}} - 1),\ \min(k_{\mathbf{B}} - 1, k_{\mathbf{C}})) + r_{\mathbf{A}} \geq R + 1. \end{aligned} \tag{1.13}$$

Then $\mathbf{\Pi}_{\mathbf{A}} = \mathbf{\Pi}_{\mathbf{C}}$ *and* $\bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{A}}^{-1}\mathbf{\Lambda}_{\mathbf{C}}^{-1}$.

Note that in Propositions 1.20 and 1.21 we do not assume that R is minimal. Neither do we assume in Proposition 1.21 that $\mathbf{\Pi}_{\mathbf{A}}$ and $\mathbf{\Pi}_{\mathbf{C}}$ are the same.

In §3 we obtain new results concerning the uniqueness of the overall CPD by combining (1.8) with results from §2.

Combining (1.8) with Proposition 1.20 we prove the following statements.

PROPOSITION 1.22. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$*. Assume that*

(i) *condition* (1.12) *holds;*

(ii) *condition* ($\mathrm{W}_{\mathrm{m}_{\mathbf{C}}}$) *holds for* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{C}$*;*

(iii) $\mathbf{A} \odot \mathbf{B}$ *has full column rank.* (C1)

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique.*

COROLLARY 1.23. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$*. Assume that*

(i) *condition* (1.12) *holds;*

(ii) *condition* ($\mathrm{U}_{\mathrm{m}_{\mathbf{C}}}$) *holds for* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique.*

COROLLARY 1.24. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$*. Assume that*

(i) *condition* (1.12) *holds;*

(ii) *condition* ($\mathrm{H}_{\mathrm{m}_{\mathbf{C}}}$) *holds for* $\mathbf{A}$ *and* $\mathbf{B}$*.*

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique.*

COROLLARY 1.25. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and* $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$*. Assume that*

(i) *condition* (1.12) *holds;*

(ii) $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B})$ *has full column rank.*

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique.*

Note that Proposition 1.15 is a special case of the results in Proposition 1.22, Corollaries 1.23–1.25 and Kruskal's Theorem 1.8. In the former, one factor matrix is assumed to have full column rank ($r_{\mathbf{C}} = R$) while in the latter this is not necessary ($r_{\mathbf{C}} = R - m_{\mathbf{C}} + 2$ with $m_{\mathbf{C}} \geq 2$). The condition on $\mathbf{C}$ is relaxed by tightening the conditions on $\mathbf{A}$ and $\mathbf{B}$. For instance, Corollary 1.23 allows $r_{\mathbf{C}} = R - m_{\mathbf{C}} + 2$ with $m := m_{\mathbf{C}} \geq 2$ by imposing (1.12) and (C_m). From scheme (1.6) we have that (C_m) implies (C_2), and hence (C_m) is more restrictive than (C_2). Scheme (1.6) further shows that Corollary 1.23 is more general than Corollaries 1.24 and 1.25. In turn, Proposition 1.22 is more general than Corollary 1.23. Note that we did not

formulate a combination of implication $(\mathrm{K_m}) \Rightarrow (\mathrm{C_m})$ (or $(\mathrm{H_m})$) from scheme (1.8) with Proposition 1.20. Such a combination leads to a result that is equivalent to Corollary 1.29 below.

Combining (1.8) with Proposition 1.21 we prove the following results.

PROPOSITION 1.26. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and let*

$$m_{\mathbf{A}} := R - r_{\mathbf{A}} + 2, \quad m_{\mathbf{B}} := R - r_{\mathbf{B}} + 2, \quad m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2. \tag{1.14}$$

Assume that at least two of the following conditions hold

(i) *condition* $(\mathrm{U}_{m_\mathbf{A}})$ *holds for* $\mathbf{B}$ *and* $\mathbf{C}$*;*
(ii) *condition* $(\mathrm{U}_{m_\mathbf{B}})$ *holds for* $\mathbf{C}$ *and* $\mathbf{A}$*;*
(iii) *condition* $(\mathrm{U}_{m_\mathbf{C}})$ *holds for* $\mathbf{A}$ *and* $\mathbf{B}$.

Then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

COROLLARY 1.27. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and consider $m_{\mathbf{A}}$, $m_{\mathbf{B}}$, and $m_{\mathbf{C}}$ defined in* (1.14)*. Assume that at least two of the following conditions hold*

(i) *condition* $(\mathrm{H}_{m_\mathbf{A}})$ *holds for* $\mathbf{B}$ *and* $\mathbf{C}$*;*
(ii) *condition* $(\mathrm{H}_{m_\mathbf{B}})$ *holds for* $\mathbf{C}$ *and* $\mathbf{A}$*;*
(iii) *condition* $(\mathrm{H}_{m_\mathbf{C}})$ *holds for* $\mathbf{A}$ *and* $\mathbf{B}$.

Then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

COROLLARY 1.28. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and consider $m_{\mathbf{A}}$, $m_{\mathbf{B}}$, and $m_{\mathbf{C}}$ defined in* (1.14)*. Let at least two of the matrices*

$$\mathcal{C}_{m_\mathbf{A}}(\mathbf{B}) \odot \mathcal{C}_{m_\mathbf{A}}(\mathbf{C}), \quad \mathcal{C}_{m_\mathbf{B}}(\mathbf{C}) \odot \mathcal{C}_{m_\mathbf{B}}(\mathbf{A}), \quad \mathcal{C}_{m_\mathbf{C}}(\mathbf{A}) \odot \mathcal{C}_{m_\mathbf{C}}(\mathbf{B}) \tag{1.15}$$

have full column rank. Then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

COROLLARY 1.29. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and let $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ coincide with $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, $(\mathbf{B}, \mathbf{C}, \mathbf{A})$, or $(\mathbf{C}, \mathbf{A}, \mathbf{B})$. If*

$$\begin{cases} k_{\mathbf{X}} + r_{\mathbf{Y}} + r_{\mathbf{Z}} & \geq 2R + 2, \\ \min(r_{\mathbf{Z}} + k_{\mathbf{Y}}, k_{\mathbf{Z}} + r_{\mathbf{Y}}) & \geq \ R + 2, \end{cases} \tag{1.16}$$

then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

COROLLARY 1.30. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ and let the following conditions hold*

$$\begin{cases} k_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} \geq \ 2R + 2, \\ r_{\mathbf{A}} + k_{\mathbf{B}} + r_{\mathbf{C}} \geq \ 2R + 2, \\ r_{\mathbf{A}} + r_{\mathbf{B}} + k_{\mathbf{C}} \geq \ 2R + 2. \end{cases} \tag{1.17}$$

Then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

Let us compare Kruskal's Theorems 1.8–1.10 with Corollaries 1.24, 1.27, 1.29, and 1.30. Elementary algebra yields that Theorem 1.9 is equivalent to Corollary 1.29. From Corollary 1.27 it follows that assumption (i) of Theorem 1.10 is redundant. We will demonstrate in Examples 3.2 and 3.3 that it is not possible to state in general which of the Corollaries 1.24 or 1.27 is more relaxed. Thus, Corollary 1.24 (obtained by combining implication $(\mathrm{H_m}) \Rightarrow (\mathrm{U_m})$ from scheme (1.8) with Proposition 1.21) is an $(\mathrm{H_m})$–type result on uniqueness that was not in [14]. Corollary 1.30 is a special case of Corollary 1.29, which is obviously more relaxed than Kruskal's well-known Theorem 1.8. Finally we note that if condition $(\mathrm{H_m})$ holds, then $r_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} \geq 2R + 2$. Thus, neither Kruskal's Theorems 1.8–1.10 nor Corollaries 1.24, 1.27, 1.29, 1.30 can be used for demonstrating the uniqueness of a PD $[\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ when $r_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} < 2R + 2$.

We did not present a result based on a combination of ($\mathrm{W_m}$)-type implications from scheme (1.8) with Proposition 1.21 because we do not have examples of cases where such conditions are more relaxed than those in Proposition 1.26.

In §4 we indicate how our results can be adapted in the case of PD symmetries.

Well-known necessary conditions for the uniqueness of the CPD are [21, p. 2079, Theorem 2], [13, p. 28], [18, p. 651]

$$\min(k_{\mathbf{A}}, k_{\mathbf{B}}, k_{\mathbf{C}}) \geq 2, \tag{1.18}$$

$$\mathbf{A} \odot \mathbf{B}, \quad \mathbf{B} \odot \mathbf{C}, \quad \mathbf{C} \odot \mathbf{A} \quad \text{have full column rank.} \tag{1.19}$$

Further, the following necessary condition was obtained in [5, Theorem 2.3]

$$(\mathrm{U_2}) \text{ holds for pairs } \quad (\mathbf{A}, \mathbf{B}), \quad (\mathbf{B}, \mathbf{C}), \quad \text{and} \quad (\mathbf{C}, \mathbf{A}). \tag{1.20}$$

It follows from scheme (1.6) that (1.20) is more restrictive than (1.18) and (1.19). Our most general condition concerning uniqueness of one factor matrix is given in Proposition 1.13. Note that in Proposition 1.13, condition (i) is more relaxed than (1.18) and condition (iii) coincides with (1.19). One may wonder whether condition (iv) in Proposition 1.13 is necessary for the uniqueness of at least one factor matrix. In §5 we show that this is not the case. We actually study an example in which CPD uniqueness can be established without ($\mathrm{W_m}$) being satisfied.

In §6 we study generic uniqueness of one factor matrix and generic CPD uniqueness. Our result on overall CPD uniqueness is the following.

PROPOSITION 1.31. *The CPD of an $I \times J \times K$ tensor of rank R is generically unique if there exist matrices $\mathbf{A}_0 \in \mathbb{F}^{I\times R}$, $\mathbf{B}_0 \in \mathbb{F}^{J\times R}$, and $\mathbf{C}_0 \in \mathbb{F}^{K\times R}$ such that at least one of the following conditions holds:*

(i) $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank, where* $m_{\mathbf{C}} = R - \min(K, R) + 2$*;*

(ii) $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{B}_0) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C}_0)$ *has full column rank, where* $m_{\mathbf{A}} = R - \min(I, R) + 2$*;*

(iii) $\mathcal{C}_{m_{\mathbf{B}}}(\mathbf{C}_0) \odot \mathcal{C}_{m_{\mathbf{B}}}(\mathbf{A}_0)$ *has full column rank, where* $m_{\mathbf{B}} = R - \min(J, R) + 2$*.*

We give several examples that illustrate the uniqueness results in the generic case.

2. Equality of PDs with common factor matrices.

In this section we assume that a tensor admits two not necessarily canonical PDs that have one or two factor matrices in common. In the latter case, the two PDs may have the columns of the common factor matrices permuted differently. We establish conditions that guarantee that the two PDs are the same.

2.1. One factor matrix in common.

In this subsection we assume that two PDs have the factor matrix $\mathbf{C}$ in common. The result that we are concerned with, is Proposition 1.20. The proof is based on the following three lemmas.

LEMMA 2.1. *For matrices $\mathbf{A}, \bar{\mathbf{A}} \in \mathbb{F}^{I\times R}$ and indices $r_1, \dots, r_n \in \{1, \dots, R\}$ define the subspaces $E_{r_1 \dots r_n}$ and $\bar{E}_{r_1 \dots r_n}$ as follows*

$$E_{r_1\dots r_n} := \operatorname{span}\{\mathbf{a}_{r_1}, \dots, \mathbf{a}_{r_n}\}, \qquad \bar{E}_{r_1\dots r_n} := \operatorname{span}\{\bar{\mathbf{a}}_{r_1}, \dots, \bar{\mathbf{a}}_{r_n}\}.$$

Assume that $k_{\mathbf{A}} \geq 2$ and that there exists $m \in \{2, \dots, k_{\mathbf{A}}\}$ such that

$$E_{r_1\dots r_{m-1}} \subseteq \bar{E}_{r_1\dots r_{m-1}} \qquad \textit{for all} \qquad 1 \leq r_1 < r_2 < \dots < r_{m-1} \leq R. \tag{2.1}$$

Then there exists a nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{A} = \bar{\mathbf{A}}\mathbf{\Lambda}$.

Proof. For $m = 2$ we have

$$\operatorname{span}\{\mathbf{a}_{r_1}\} = E_{r_1} \subseteq \bar{E}_{r_1} = \operatorname{span}\{\bar{\mathbf{a}}_{r_1}\}, \qquad \text{for all } 1 \leq r_1 \leq R, \tag{2.2}$$

such that the Lemma trivially holds. For $m \geq 3$ we arrive at (2.2) by downward induction on $l = m, m-1, \dots, 3$. Assuming that

$$E_{r_1 \dots r_{l-1}} \subseteq \bar{E}_{r_1 \dots r_{l-1}} \qquad \text{for all} \qquad 1 \leq r_1 < r_2 < \dots < r_{l-1} \leq R, \tag{2.3}$$

we show that

$$E_{r_1 \dots r_{l-2}} \subseteq \bar{E}_{r_1 \dots r_{l-2}} \qquad \text{for all} \qquad 1 \leq r_1 < r_2 < \dots < r_{l-2} \leq R.$$

Assume $r_1, r_2, \dots, r_{l-2}$ fixed and let $i, j \in \{1, \dots, R\} \setminus \{r_1, \dots, r_{l-2}\}$, with $i \neq j$. Since $l \leq m \leq k_{\mathbf{A}}$, we have that $\dim E_{r_1,\dots,r_{l-2},i,j} = l$. Because

$$\begin{aligned} l = \dim E_{r_1,\dots,r_{l-2},i,j} &\leq \dim \operatorname{span}\{E_{r_1,\dots,r_{l-2},i},\ E_{r_1,\dots,r_{l-2},j}\} \\ &\overset{(2.3)}{\leq} \dim \operatorname{span}\{\bar{E}_{r_1,\dots,r_{l-2},i},\ \bar{E}_{r_1,\dots,r_{l-2},j}\} \end{aligned}$$

we have

$$\bar{E}_{r_1,\dots,r_{l-2},i} \neq \bar{E}_{r_1,\dots,r_{l-2},j}. \tag{2.4}$$

Therefore,

$$\begin{aligned} E_{r_1,\dots,r_{l-2}} &\subseteq \left(E_{r_1,\dots,r_{l-2},i} \cap E_{r_1,\dots,r_{l-2},j}\right) \\ &\overset{(2.3)}{\subseteq} \left(\bar{E}_{r_1,\dots,r_{l-2},i} \cap \bar{E}_{r_1,\dots,r_{l-2},j}\right) \overset{(2.4)}{=} \bar{E}_{r_1,\dots,r_{l-2}}. \end{aligned}$$

The induction follows. To conclude the proof, we note that $\mathbf{\Lambda}$ is nonsingular since $k_{\mathbf{A}} \geq 2$. ∎

LEMMA 2.2. *Let* $\mathbf{C} \in \mathbb{F}^{K \times R}$ *and consider* m *such that* $m \leq k_{\mathbf{C}}$. *Then for any set of distinct indices* $\mathcal{I} = \{i_1, \dots, i_{m-1}\} \subseteq \{1, \dots, R\}$ *there exists a vector* $\mathbf{x} \in \mathbb{F}^K$ *such that*

$$\mathbf{x}^T \mathbf{c}_i = 0 \ \text{ for } \ i \in \mathcal{I} \ \text{ and } \ \mathbf{x}^T \mathbf{c}_i \neq 0 \text{ for } i \in \mathcal{I}^c := \{1, \dots, R\} \setminus \mathcal{I}. \tag{2.5}$$

Proof. Let $\mathbf{C}_{\mathcal{I}} \in \mathbb{F}^{K \times (m-1)}$ and $\mathbf{C}_{\mathcal{I}^c} \in \mathbb{F}^{K \times (R-m+1)}$ contain the columns of $\mathbf{C}$ indexed by $\mathcal{I}$ and $\mathcal{I}^c$, respectively, and let the columns of $\mathbf{C}_{\mathcal{I}}^{\perp} \in \mathbb{F}^{K \times (K-m+1)}$ form a basis for the orthogonal complement of range$(\mathbf{C}_{\mathcal{I}})$. The matrix $(\mathbf{C}_{\mathcal{I}}^{\perp})^H \mathbf{C}_{\mathcal{I}^c}$ cannot have a zero column, otherwise the corresponding column of $\mathbf{C}_{\mathcal{I}^c}$ would be in range$(\mathbf{C}_{\mathcal{I}})$, which would be a contradiction with $k_{\mathbf{C}} \geq m$. We conclude that (2.5) holds for $\mathbf{x} = (\mathbf{C}_{\mathcal{I}}^{\perp} \mathbf{y})^*$, with $\mathbf{y} \in \mathbb{F}^{K-m+1}$ generic. ∎

LEMMA 2.3. *Let* $\mathbf{P}$ *be an* $R \times R$ *permutation matrix. Then for any vector* $\lambda \in \mathbb{F}^R$,

$$\operatorname{Diag}(\mathbf{\Pi}\lambda)\mathbf{\Pi} = \mathbf{\Pi}\operatorname{Diag}(\lambda). \tag{2.6}$$

Proof. The lemma follows directly from the definition of permutation matrix. ∎

We are now ready to prove Proposition 1.20.

Proof. Let $\widehat{\mathbf{A}} := \bar{\mathbf{A}}\mathbf{\Pi}^T$ and $\widehat{\mathbf{B}} := \bar{\mathbf{B}}\mathbf{\Lambda}_{\mathbf{C}}^{-1}\mathbf{\Pi}^T$. Then

$$\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}]_R = [\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \mathbf{C}]_R. \tag{2.7}$$

We show that the columns of $\mathbf{A}$ and $\mathbf{B}$ coincide up to scaling with the corresponding columns of $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$, respectively. Consider indices $i_1, \dots, i_{R-k_{\mathbf{C}}+1}$ such that $1 \leq$

$i_1 < \cdots < i_{R-k_{\mathbf{C}}+1} \leq R$. Let $m := k_{\mathbf{C}}$ and let $\mathcal{I} := \{1, \ldots, R\} \setminus \{i_1, \ldots, i_{R-k_{\mathbf{C}}+1}\}$. From Lemma 2.2 it follows that there exists a vector $\mathbf{x} \in \mathbb{F}^K$ such that

$$\mathbf{x}^T\mathbf{c}_i = 0 \quad \text{for} \quad i \in \mathcal{I} \quad \text{and} \; \mathbf{x}^T\mathbf{c}_i \neq 0 \quad \text{for} \quad i \in \mathcal{I}^{\mathrm{c}} = \{i_1, \ldots, i_{R-k_{\mathbf{C}}+1}\}.$$

Let $\mathbf{d} = \begin{bmatrix}\mathbf{x}^T\mathbf{c}_{i_1} & \ldots & \mathbf{x}^T\mathbf{c}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}^T$. Then $(\mathbf{A}\odot\mathbf{B})\mathbf{C}^T\mathbf{x} = (\widehat{\mathbf{A}}\odot\widehat{\mathbf{B}})\mathbf{C}^T\mathbf{x}$ is equivalent to

$$\begin{aligned}&\left(\begin{bmatrix}\mathbf{a}_{i_1} & \ldots & \mathbf{a}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix} \odot \begin{bmatrix}\mathbf{b}_{i_1} & \ldots & \mathbf{b}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}\right)\mathbf{d} = \\ &\left(\begin{bmatrix}\widehat{\mathbf{a}}_{i_1} & \ldots & \widehat{\mathbf{a}}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix} \odot \begin{bmatrix}\widehat{\mathbf{b}}_{i_1} & \ldots & \widehat{\mathbf{b}}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}\right)\mathbf{d},\end{aligned}$$

which may be expressed as

$$\begin{aligned}&\begin{bmatrix}\mathbf{a}_{i_1} & \ldots & \mathbf{a}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}\operatorname{Diag}(\mathbf{d})\begin{bmatrix}\mathbf{b}_{i_1} & \ldots & \mathbf{b}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}^T \\ &\qquad = \begin{bmatrix}\widehat{\mathbf{a}}_{i_1} & \ldots & \widehat{\mathbf{a}}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}\operatorname{Diag}(\mathbf{d})\begin{bmatrix}\widehat{\mathbf{b}}_{i_1} & \ldots & \widehat{\mathbf{b}}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}^T.\end{aligned}$$

By (1.12), $\min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq R - k_{\mathbf{C}} + 1$. Hence, the matrices $\begin{bmatrix}\mathbf{a}_{i_1} & \ldots & \mathbf{a}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{b}_{i_1} & \ldots & \mathbf{b}_{i_{R-k_{\mathbf{C}}+1}}\end{bmatrix}$ have full column rank. Since by construction the vector $\mathbf{d}$ has only nonzero components, it follows that

$$\begin{aligned}\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_{R-k_{\mathbf{C}}+1}} &\in \operatorname{span}\{\widehat{\mathbf{a}}_{i_1}, \ldots, \widehat{\mathbf{a}}_{i_{R-k_{\mathbf{C}}+1}}\}, \\ \mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_{R-k_{\mathbf{C}}+1}} &\in \operatorname{span}\{\widehat{\mathbf{b}}_{i_1}, \ldots, \widehat{\mathbf{b}}_{i_{R-k_{\mathbf{C}}+1}}\}.\end{aligned}$$

By (1.12), $\max(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m := R - k_{\mathbf{C}} + 2 \geq 2$. Without loss of generality we confine ourselves to the case $k_{\mathbf{A}} \geq m$. Then, by Lemma 2.1, there exists a nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{A} = \widehat{\mathbf{A}}\mathbf{\Lambda}$. Denoting $\lambda_{\mathbf{A}} := \mathbf{\Pi}^T\operatorname{diag}(\mathbf{\Lambda}^{-1})$ and $\mathbf{\Lambda}_{\mathbf{A}} = \operatorname{Diag}(\lambda_{\mathbf{A}})$ and applying Lemma 2.3, we have

$$\bar{\mathbf{A}} = \widehat{\mathbf{A}}\mathbf{\Pi} = \mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{\Pi} = \mathbf{A}\operatorname{Diag}(\mathbf{\Pi}\lambda_{\mathbf{A}})\mathbf{\Pi} = \mathbf{A}\mathbf{\Pi}\operatorname{Diag}(\lambda_{\mathbf{A}}) = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}}.$$

It follows from (2.7) and (1.2) that

$$(\mathbf{C}\odot\mathbf{A})\mathbf{B}^T = (\mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}\odot\bar{\mathbf{A}})\bar{\mathbf{B}}^T = (\mathbf{C}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}\odot\mathbf{A}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}})\bar{\mathbf{B}}^T = (\mathbf{C}\odot\mathbf{A})\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}\mathbf{\Lambda}_{\mathbf{A}}\bar{\mathbf{B}}^T.$$

Since $k_{\mathbf{A}} \geq R - k_{\mathbf{C}} + 2$, it follows that condition (K1) holds for the matrices $\mathbf{A}$ and $\mathbf{C}$. From Proposition 1.12 (1) it follows that the matrix $\mathbf{C}\odot\mathbf{A}$ has full column rank. Hence, $\mathbf{B}^T = \mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{C}}\mathbf{\Lambda}_{\mathbf{A}}\bar{\mathbf{B}}^T$, i.e., $\bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{A}}^{-1}\mathbf{\Lambda}_{\mathbf{C}}^{-1} =: \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda}_{\mathbf{B}}$. $\square$

EXAMPLE 2.4. *Consider the* $2\times 3\times 3$ *tensor given by* $\mathcal{T} = [\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}]_3$, *where*

$$\widehat{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 3\end{bmatrix}, \quad \widehat{\mathbf{B}} = \begin{bmatrix} 6 & 12 & 2 \\ 3 & 4 & -1 \\ 4 & 6 & -4\end{bmatrix}, \quad \widehat{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}.$$

Since $k_{\widehat{\mathbf{A}}} + k_{\widehat{\mathbf{B}}} + k_{\widehat{\mathbf{C}}} = 2 + 3 + 3 \geq 2\times 3 + 2$, *it follows from Theorem 1.8 that* $r_{\mathcal{T}} = 3$ *and that the CPD of* $\mathcal{T}$ *is unique.*

Increasing the number of terms, we also have $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_4$ *for*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 6 & -6 & -3 & -2 \\ 12 & -24 & -8 & -6 \\ 2 & 6 & -3 & -6\end{bmatrix}.$$

Since $k_{\mathbf{A}} = 2$ and $k_{\mathbf{B}} = k_{\mathbf{C}} = 3$, condition (1.12) *holds. Hence, by Proposition 1.20, if $\mathcal{T} = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_4$ and $\bar{\mathbf{C}} = \mathbf{C}$, then there exists a nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Lambda}$ and $\bar{\mathbf{B}} = \mathbf{B}\mathbf{\Lambda}^{-1}$.*

The following condition is also satisfied:

$$\max(\min(k_{\mathbf{A}}, k_{\mathbf{C}} - 1),\ \min(k_{\mathbf{A}} - 1, k_{\mathbf{C}})) + k_{\mathbf{B}} \geq R + 1.$$

By symmetry, we have from Proposition 1.20 that, if $\mathcal{T} = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_4$ and $\bar{\mathbf{B}} = \mathbf{B}$, then there exists a nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Lambda}$ and $\bar{\mathbf{C}} = \mathbf{C}\mathbf{\Lambda}^{-1}$.

Finally, we show that the inequality of condition (1.12) *is sharp. We have*

$$\max(\min(k_{\mathbf{B}}, k_{\mathbf{C}} - 1),\ \min(k_{\mathbf{B}} - 1, k_{\mathbf{C}})) + k_{\mathbf{A}} = R < R + 1.$$

One can verify that $\mathcal{T} = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_4$ with $\bar{\mathbf{A}} = \mathbf{A}$ and with $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ given by

$$\bar{\mathbf{B}} = \begin{bmatrix} 6 & 12 & 2 \\ 3 & 4 & -1 \\ 4 & 6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4/3 & 3/2 \\ 1 & -3 & 3 & 9 \end{bmatrix},$$

$$\bar{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 1/\beta \end{bmatrix} \begin{bmatrix} 6 & -6 & -3 & -2 \\ -24/5 & 48/5 & 16/5 & 12/5 \\ 2/15 & 2/5 & -1/5 & -2/5 \end{bmatrix},$$

for arbitrary nonzero α and β. Hence, there exist infinitely many PDs $\mathcal{T} = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_4$ with $\bar{\mathbf{A}} = \mathbf{A}$; the columns of $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ are only proportional to the columns of $\mathbf{B}$ and $\mathbf{C}$, respectively, for $\alpha = -2/5$ and $\beta = 1/15$. We conclude that the inequality of condition (1.12) *is sharp.*

2.2. Two factor matrices in common. In this subsection we assume that two PDs have the factor matrices $\mathbf{A}$ and $\mathbf{C}$ in common. We do not assume however that in the two PDs the columns of these matrices are permuted in the same manner. The result that we are concerned with, is Proposition 1.21.

Proof. Without loss of generality, we confine ourselves to the case

$$k_{\mathbf{C}} \geq 2 \quad \text{and} \quad \min(k_{\mathbf{A}} - 1, k_{\mathbf{B}}) + r_{\mathbf{C}} \geq R + 1. \tag{2.8}$$

We set for brevity $r := r_{\mathbf{C}}$. Denoting $\mathbf{\Pi} = \mathbf{\Pi}_{\mathbf{A}}\mathbf{\Pi}_{\mathbf{C}}^T$ and $\widehat{\mathbf{B}} = \bar{\mathbf{B}}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{C}}\mathbf{\Pi}_{\mathbf{C}}^T$, we have $[\mathbf{A}\mathbf{\Pi}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{A}}, \bar{\mathbf{B}}, \mathbf{C}\mathbf{\Pi}_{\mathbf{C}}\mathbf{\Lambda}_{\mathbf{C}}]_R = [\mathbf{A}\mathbf{\Pi}_{\mathbf{A}}\mathbf{\Pi}_{\mathbf{C}}^T, \bar{\mathbf{B}}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{C}}\mathbf{\Pi}_{\mathbf{C}}^T, \mathbf{C}]_R = [\mathbf{A}\mathbf{\Pi}, \widehat{\mathbf{B}}, \mathbf{C}]_R$. We will show that, under (2.8), $[\mathbf{A}, \mathbf{B}, \mathbf{C}]_R = [\mathbf{A}\mathbf{\Pi}, \widehat{\mathbf{B}}, \mathbf{C}]_R$ implies that $\mathbf{\Pi} = \mathbf{I}_R$. This, in turn, immediately implies that $\mathbf{\Pi}_{\mathbf{A}} = \mathbf{\Pi}_{\mathbf{C}}$ and $\bar{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}_{\mathbf{A}}\mathbf{\Lambda}_{\mathbf{A}}^{-1}\mathbf{\Lambda}_{\mathbf{C}}^{-1}$.

(i) Let us fix integers $i_1, \dots, i_r$ such that the columns $\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_r}$ form a basis of range$(\mathbf{C})$ and let us set $\{j_1, \dots, j_{R-r}\} := \{1, \dots, R\} \setminus \{i_1, \dots, i_r\}$. Let $\mathbf{X} \in \mathbb{F}^{K \times r}$, denote a right inverse of $\begin{bmatrix} \mathbf{c}_{i_1} & \dots & \mathbf{c}_{i_r} \end{bmatrix}^T$, i.e., $\begin{bmatrix} \mathbf{c}_{i_1} & \dots & \mathbf{c}_{i_r} \end{bmatrix}^T \mathbf{X} = \mathbf{I}_r$. Define the subspaces $E, E_{i_k} \subseteq \mathbb{F}^R$ as follows:

$$\begin{aligned} E &= \operatorname{span}\{\mathbf{e}_{j_1}^R, \dots \mathbf{e}_{j_{R-r}}^R\}, \\ E_{i_k} &= \operatorname{span}\{\mathbf{e}_l^R :\ \mathbf{c}_l^T \mathbf{x}_k \neq 0,\ l \in \{j_1, \dots, j_{R-r}\}\}, \qquad k \in \{1, \dots, r\}. \end{aligned}$$

By construction, $E_{i_k} \subseteq E$ and $\mathbf{e}_{i_l}^R \notin E_{i_k}$, $k, l \in \{1, \dots, r\}$.

(ii) Let us show that $\mathbf{\Pi}\text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\} = \text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\}$ for all $k \in \{1,\dots,r\}$. Let us fix $k \in \{1,\dots,r\}$. Assume that $\mathbf{C}^T\mathbf{x}_k$ has nonzero entries at positions $k_1,\dots,k_L$. Denote these entries by $\alpha_1,\dots,\alpha_L$. From the definition of $\mathbf{X}$ and E_{i_k} it follows that $L \le R-r+1$ and $\text{span}\{\mathbf{e}_{k_1}^R,\dots,\mathbf{e}_{k_L}^R\} = \text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\}$.
Define $\mathbf{P}_k = \begin{bmatrix}\mathbf{e}_{k_1}^R & \dots & \mathbf{e}_{k_L}^R\end{bmatrix}$. Then we have

$$\mathbf{P}_k\mathbf{P}_k^T\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{P}_k\mathbf{P}_k^T = \text{Diag}(\mathbf{C}^T\mathbf{x}_k), \tag{2.9}$$

$$\mathbf{P}_k^T\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{P}_k = \text{Diag}(\begin{bmatrix}\alpha_1 & \dots & \alpha_L\end{bmatrix}). \tag{2.10}$$

Further, $[\mathbf{A},\mathbf{B},\mathbf{C}]_R = [\mathbf{A\Pi},\widehat{\mathbf{B}},\mathbf{C}]_R$ implies that

$$\mathbf{A}\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{B}^T = \mathbf{A\Pi}\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\widehat{\mathbf{B}}^T. \tag{2.11}$$

Using (2.9)–(2.11), we obtain

$$\begin{aligned}\mathbf{A}\mathbf{P}_k\text{Diag}(\begin{bmatrix}\alpha_1 & \dots & \alpha_L\end{bmatrix})\mathbf{P}_k^T\mathbf{B}^T &= \mathbf{A}\mathbf{P}_k\mathbf{P}_k^T\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{P}_k\mathbf{P}_k^T\mathbf{B}^T\\ &= \mathbf{A}\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{B}^T\\ &= \mathbf{A\Pi}\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\widehat{\mathbf{B}}^T\\ &= \mathbf{A\Pi}\mathbf{P}_k\mathbf{P}_k^T\text{Diag}(\mathbf{C}^T\mathbf{x}_k)\mathbf{P}_k\mathbf{P}_k^T\widehat{\mathbf{B}}^T\\ &= \mathbf{A\Pi}\mathbf{P}_k\text{Diag}(\begin{bmatrix}\alpha_1 & \dots & \alpha_L\end{bmatrix})\mathbf{P}_k^T\widehat{\mathbf{B}}^T. \end{aligned} \tag{2.12}$$

Note that $\mathbf{B}\mathbf{P}_k = \begin{bmatrix}\mathbf{b}_{k_1} & \dots & \mathbf{b}_{k_L}\end{bmatrix}$. Since, by (2.8), $k_{\mathbf{B}} \ge R-r+1 \ge L$, it follows that the matrix $\mathbf{P}_k^T\widehat{\mathbf{B}}^T$ has full row rank. Further noting that $\mathbf{A}\mathbf{P}_k = \begin{bmatrix}\mathbf{a}_{k_1} & \dots & \mathbf{a}_{k_L}\end{bmatrix}$ and $\mathbf{A\Pi}\mathbf{P}_k = \begin{bmatrix}(\mathbf{A\Pi})_{k_1} & \dots & (\mathbf{A\Pi})_{k_L}\end{bmatrix}$, we obtain from (2.12) that

$$\text{span}\{\mathbf{a}_{k_1},\dots,\mathbf{a}_{k_L}\} \subseteq \text{span}\{(\mathbf{A\Pi})_{k_1},\dots,(\mathbf{A\Pi})_{k_L}\}. \tag{2.13}$$

Since, by (2.8), $k_{\mathbf{A}} \ge R-r+2 \ge L+1$, (2.13) is only possible if $\mathbf{\Pi}\text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\} = \text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\}$.

(iii) Let us show that $\mathbf{\Pi}E = E$. Let us fix $j \in \{j_1,\dots,j_{R-r}\}$. From $\mathbf{X}^T\mathbf{c}_{i_k} = \mathbf{e}_k^r$ for $k \in \{1,\dots,r\}$, the fact that the vectors $\mathbf{c}_{i_1},\dots,\mathbf{c}_{i_r}$ form a basis of range$(\mathbf{C})$, and $k_{\mathbf{C}} \ge 2$, it follows that the vector $\mathbf{X}^T\mathbf{c}_j$ has at least two nonzero components, say, the m-th and n-th component. Since $\mathbf{c}_j^T\mathbf{x}_m \ne 0$ and $\mathbf{c}_j^T\mathbf{x}_n \ne 0$, we have $\mathbf{e}_j^R \in E_{i_m} \cap E_{i_n}$. From the preceding steps we have

$$\begin{aligned}\mathbf{\Pi}\mathbf{e}_j^R \in \mathbf{\Pi}(E_{i_m}\cap E_{i_n}) &\overset{\text{(i)}}{=} \mathbf{\Pi}\left(\text{span}\{E_{i_m},\mathbf{e}_{i_m}^R\}\cap\text{span}\{E_{i_n},\mathbf{e}_{i_n}^R\}\right)\\ &\overset{\text{(ii)}}{\subseteq} \text{span}\{E_{i_m},\mathbf{e}_{i_m}^R\}\cap\text{span}\{E_{i_n},\mathbf{e}_{i_n}^R\} \overset{\text{(i)}}{=} E_{i_m}\cap E_{i_n} \subseteq E.\end{aligned}$$

Since this holds true for any index $j \in \{j_1,\dots,j_{R-r}\}$, it follows that $\mathbf{\Pi}E = E$.

(iv) Let us show that $\mathbf{\Pi}\mathbf{e}_{i_k}^R = \mathbf{e}_{i_k}^R$ for all $k \in \{1,\dots,r\}$. From the preceding steps we have

$$\mathbf{\Pi}E_{i_k} \overset{\text{(i)}}{=} \mathbf{\Pi}\left(\text{span}\{E_{i_k},\mathbf{e}_{i_k}^R\}\cap E\right) \overset{\text{(ii), (iii)}}{\subseteq} \text{span}\{E_{i_k},\mathbf{e}_{i_k}^R\}\cap E \overset{\text{(i)}}{=} E_{i_k}.$$

On the other hand, we have from step (iii) that $\mathbf{\Pi}\text{span}\{E_{i_k}, \mathbf{e}_{i_k}^R\} = \{E_{i_k}, \mathbf{e}_{i_k}^R\}$, with, as shown in step (i), $\mathbf{e}_{i_k}^R \notin E_{i_k}$. It follows that $\mathbf{\Pi}\mathbf{e}_{i_k}^R = \mathbf{e}_{i_k}^R$ for all $k \in \{1,\dots,r\}$.

(v) We have so far shown that, if the columns $\mathbf{c}_{i_1},\dots,\mathbf{c}_{i_r}$ form a basis of range$(\mathbf{C})$, then $\mathbf{\Pi}\begin{bmatrix}\mathbf{e}_{i_1}^R & \dots \mathbf{e}_{i_r}^R\end{bmatrix} = \begin{bmatrix}\mathbf{e}_{i_1}^R & \dots \mathbf{e}_{i_r}^R\end{bmatrix}$. To complete the proof of the overall equality $\mathbf{\Pi} = \mathbf{I}_R$, it suffices to note that a basis of range$(\mathbf{C})$ can be constructed starting from any column of $\mathbf{C}$.
□

3. Overall CPD uniqueness. In Proposition 1.22 and Corollaries 1.23–1.25 overall CPD uniqueness is derived from uniqueness of one factor matrix, where the latter is guaranteed by Proposition 1.20. In Proposition 1.26 and Corollaries 1.28–1.30 overall CPD is derived from uniqueness of two factor matrices, where the latter is guaranteed by Proposition 1.21. We illustrate our results with some examples.

Proof of Proposition 1.22. By (1.12), $k_{\mathbf{C}} \geq 1$ and $\min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m_{\mathbf{C}} - 1$. Hence, by Proposition 1.14, $r_{\mathcal{T}} = R$ and the third factor matrix of $\mathcal{T}$ is unique. The result now follows from Proposition 1.20. ∎

Proof of Corollary 1.23. From Proposition 1.12 (3) it follows that (W$\mathrm{m}_{\mathbf{C}}$) holds for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Since (U1) is equivalent to (C1), it follows from Proposition 1.12 (7) that $\mathbf{A} \odot \mathbf{B}$ has full column rank. The result now follows from Proposition 1.22. ∎

Proof of Corollaries 1.24 and 1.25. By Proposition 1.12 (2), both (H$\mathrm{m}_{\mathbf{C}}$) and (C$\mathrm{m}_{\mathbf{C}}$) imply (U$\mathrm{m}_{\mathbf{C}}$). The result now follows from Corollary 1.23. ∎

Proof of Proposition 1.26. Without loss of generality we assume that (i) and (iii) hold. By Proposition 1.12 (9),

$$\min(k_{\mathbf{B}}, k_{\mathbf{C}}) \geq m_{\mathbf{A}} \geq 2, \qquad \min(k_{\mathbf{A}}, k_{\mathbf{B}}) \geq m_{\mathbf{C}} \geq 2. \tag{3.1}$$

It follows from Proposition 1.14 that $r_{\mathcal{T}} = R$ and that the first and third factor matrices of the tensor $\mathcal{T}$ are unique. One can easily check that (3.1) implies (1.13). Hence, by Proposition 1.21, the CPD of $\mathcal{T}$ is unique. ∎

Proof of Corollary 1.27. Without loss of generality we assume that (ii) and (iii) hold. From Proposition 1.12 (2) it follows that (ii) and (iii) in Proposition 1.26 also hold. Hence, by Proposition 1.26, $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T}$ is unique. ∎

Proof of Corollary 1.28. By Proposition 1.12 (2), if two of the matrices in (1.15) have full column rank, then at least two of conditions (i)–(iii) in Proposition 1.26 hold. Hence, by Proposition 1.26, $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T}$ is unique. ∎

Proof of Corollary 1.29. Without loss of generality we assume that $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{B}, \mathbf{C}, \mathbf{A})$. Then,

$$\begin{cases} \begin{cases} k_{\mathbf{B}} + r_{\mathbf{A}} + r_{\mathbf{C}} & \geq 2R + 2, \\ k_{\mathbf{A}} + r_{\mathbf{C}} & \geq R + 2, \end{cases} \\ \begin{cases} k_{\mathbf{B}} + r_{\mathbf{A}} + r_{\mathbf{C}} & \geq 2R + 2, \\ r_{\mathbf{A}} + k_{\mathbf{C}} & \geq R + 2, \end{cases} \end{cases} \Rightarrow \begin{cases} (\mathrm{Km}_{\mathbf{A}}) \ \text{ holds for } \ \mathbf{B} \ \text{ and } \ \mathbf{C}, \\ (\mathrm{Km}_{\mathbf{C}}) \ \text{ holds for } \ \mathbf{A} \ \text{ and } \ \mathbf{B}, \end{cases}$$

where $m_{\mathbf{A}} = R - r_{\mathbf{A}} + 2$ and $m_{\mathbf{C}} = R - r_{\mathbf{C}} + 2$. From Proposition 1.12 (1) it follows that the matrices $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{B}) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C})$ and $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B})$ have full column rank. Hence, by Corollary 1.28, $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T}$ is unique. ∎

Proof of Corollary 1.30. It can be easily checked that all conditions of Corollary 1.29 hold. Hence, $r_{\mathcal{T}} = R$ and the CPD of $\mathcal{T}$ is unique. ∎

EXAMPLE 3.1. *Consider a $5 \times 5 \times 5$ tensor given by the PD $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_6$, where the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{5 \times 6}$ satisfy*

$$r_{\mathbf{A}} = r_{\mathbf{B}} = r_{\mathbf{C}} = 5, \qquad k_{\mathbf{A}} = k_{\mathbf{B}} = k_{\mathbf{C}} = 4.$$

For instance, consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

where $*$ denotes arbitrary nonzero entries. Then Kruskal's condition (1.4) *does not hold. On the other hand, the conditions of Corollary 1.29 are satisfied. Hence, the PD of $\mathcal{T}$ is canonical and unique.*

EXAMPLE 3.2. *Consider the $4 \times 4 \times 4$ tensor given by the PD $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_5$, where*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We have

$$r_{\mathbf{A}} = r_{\mathbf{B}} = r_{\mathbf{C}} = 4, \qquad k_{\mathbf{A}} = k_{\mathbf{B}} = k_{\mathbf{C}} = 3, \qquad m_{\mathbf{A}} = m_{\mathbf{B}} = m_{\mathbf{C}} = 3.$$

Hence, Kruskal's condition (1.4) *does not hold. Moreover, condition* (K3) *does not hold for* $(\mathbf{A}, \mathbf{B})$, $(\mathbf{C}, \mathbf{A})$, *nor* $(\mathbf{B}, \mathbf{C})$. *Hence, the conditions of Corollary 1.29 are not satisfied. On the other hand, we have*

$$\begin{aligned}
&\mathcal{C}_3(\mathbf{A}) \odot \mathcal{C}_3(\mathbf{B}) \\
&\qquad = \left[\mathbf{e}_1^{16} \quad \mathbf{e}_6^{16} \quad \mathbf{e}_2^{16} \quad \mathbf{e}_{11}^{16} \quad \mathbf{e}_{1,-3}^{16} \quad \mathbf{e}_{6,10}^{16} \quad \mathbf{e}_{16}^{16} \quad \mathbf{e}_{1,4}^{16} \quad \mathbf{e}_{6,-14}^{16} \quad \mathbf{e}_{11,12,15,16}^{16}\right], \\
&\mathcal{C}_3(\mathbf{C}) \odot \mathcal{C}_3(\mathbf{A}) \\
&\qquad = \left[\mathbf{e}_1^{16} \quad \mathbf{e}_6^{16} \quad \mathbf{e}_{1,5}^{16} \quad \mathbf{e}_{11}^{16} \quad -\mathbf{e}_9^{16} \quad \mathbf{e}_{10,11}^{16} \quad \mathbf{e}_{16}^{16} \quad \mathbf{e}_{1,13}^{16} \quad \mathbf{e}_{6,16,-8,-14}^{16} \quad \mathbf{e}_{11,12}^{16}\right], \\
&\mathcal{C}_3(\mathbf{B}) \odot \mathcal{C}_3(\mathbf{C}) \\
&\qquad = \left[\mathbf{e}_1^{16} \quad \mathbf{e}_6^{16} \quad \mathbf{e}_{5,6}^{16} \quad \mathbf{e}_{11}^{16} \quad \mathbf{e}_{11,-3}^{16} \quad \mathbf{e}_7^{16} \quad \mathbf{e}_{16}^{16} \quad \mathbf{e}_{1,4,13,16}^{16} \quad \mathbf{e}_{6,-8}^{16} \quad \mathbf{e}_{11,15}^{16}\right],
\end{aligned}$$

where

$$\mathbf{e}_{i,\pm j}^{16} := \mathbf{e}_i^{16} \pm \mathbf{e}_j^{16}, \quad \mathbf{e}_{i,j,\pm k,\pm l}^{16} := \mathbf{e}_i^{16} + \mathbf{e}_j^{16} \pm \mathbf{e}_k^{16} \pm \mathbf{e}_l^{16}, \quad i, j, k, l \in \{1, \dots, 16\}.$$

It is easy to check that the matrices $\mathcal{C}_3(\mathbf{A}) \odot \mathcal{C}_3(\mathbf{B})$, $\mathcal{C}_3(\mathbf{C}) \odot \mathcal{C}_3(\mathbf{A})$ *and* $\mathcal{C}_3(\mathbf{B}) \odot \mathcal{C}_3(\mathbf{C})$ *have full column rank. Hence, by Corollary 1.28, the PD is canonical and unique.*

One can easily verify that $H_{\mathbf{AB}}(\delta) = H_{\mathbf{BC}}(\delta) = H_{\mathbf{CA}}(\delta) = \min(\delta, 3)$. *Hence the uniqueness of the CPD follows also from Corollary 1.27.*

Note that, since condition (1.12) *does not hold, the result does not follow from Proposition 1.22 and its Corollaries 1.23–1.25.*

EXAMPLE 3.3. *Consider the $5 \times 5 \times 8$ tensor given by the PD $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_8$, where*

$$\mathbf{A} = \begin{bmatrix} \widehat{\mathbf{A}} \\ (\mathbf{e}_1^8)^T \end{bmatrix} \in \mathbb{F}^{5 \times 8}, \qquad \mathbf{B} = \begin{bmatrix} \widehat{\mathbf{B}} \\ (\mathbf{e}_8^8)^T \end{bmatrix} \in \mathbb{F}^{5 \times 8}, \qquad \mathbf{C} = \mathbf{I}_8$$

and $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ are 4×8 matrices such that $k_{\widehat{\mathbf{A}}} = k_{\widehat{\mathbf{B}}} = 4$. We have $r_{\mathbf{A}} = r_{\mathbf{B}} = 5$, $k_{\mathbf{A}} = k_{\mathbf{B}} = 4$, and $r_{\mathbf{C}} = k_{\mathbf{C}} = 8$. One can easily check that

$$H_{\mathbf{AB}}(\delta) = \begin{cases} \delta, & 1 \le \delta \le 4, \\ 3, & \delta = 5, \\ 2, & 6 \le \delta \le 8 \end{cases} \ge \min(\delta, 8 - 8 + 2)$$

and that condition (1.12) *holds. Hence, by Corollary 1.24, the PD is canonical and unique. On the other hand,* $H_{\mathbf{BC}}(\delta) = H_{\mathbf{CA}}(\delta) = 4 < \min(\delta, 8-5+2)$ *for* $\delta = 5$. *Hence, the result does not follow from Corollary 1.27.*

EXAMPLE 3.4. *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 2\\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 3\\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}, \qquad \mathbf{C} = \mathbf{I}_5.$$

It has already been shown in [17] that the CPD of the tensor $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_5$ *is unique. We give a shorter proof, based on Corollary 1.23. It is easy to verify that*

$$\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B}) = \begin{bmatrix} 1 & 0 & 1 & 6 & 0 & 1 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 10 & 0 & 0 & 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 0 & 0 & -1 & -5 & 0 & 0 & 2\\ 0 & 0 & 1 & 9 & 0 & 0 & 0 & 0 & 0 & 4\\ 0 & 1 & 1 & 15 & 0 & 0 & 0 & 1 & 1 & 8\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 0 & 0 & -1 & -3 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4\\ 0 & 0 & 0 & 0 & 1 & 1 & 15 & 1 & 6 & 2 \end{bmatrix},$$

$$\ker(\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})) = \operatorname{span}\{\begin{bmatrix} 0 & 0 & -4 & 0 & 0 & 2 & 0 & -4 & 0 & -1 \end{bmatrix}^T\}.$$

If $\mathbf{d} \in \mathbb{C}^5$ *is such that* $\operatorname{diag}(\mathcal{C}_2(\operatorname{Diag}(\mathbf{d}))) \in \ker(\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B}))$, *we have*

$$\begin{aligned} &d_1d_2 = 0, && d_2d_3 = 0, && d_3d_4 = -4c, && d_4d_5 = -c.\\ &d_1d_3 = 0, && d_2d_4 = 2c, && d_3d_5 = 0, &&\\ &d_1d_4 = -4c, && d_2d_5 = 0, && &&\\ &d_1d_5 = 0. && && && \end{aligned}$$

One can check that this set of equations only has a solution if $c = 0$, *in which case* $\mathbf{d} = \mathbf{0}$. *Hence, by Corollary 1.23, the PD is canonical and unique. Note that, since* $m_{\mathbf{A}} = m_{\mathbf{B}} = 5-3+2 = 4$, *the* $m_{\mathbf{A}}$*-th compound matrix of* $\mathbf{A}$ *and the* $m_{\mathbf{B}}$*-th compound matrix of* $\mathbf{B}$ *are not defined. Hence, the uniqueness of the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *does not follow from Proposition 1.26.*

EXAMPLE 3.5. *Experiments indicate that for random* 7×10 *matrices* $\mathbf{A}$ *and* $\mathbf{B}$, *the matrix* $\mathbf{A} \odot \mathbf{B}$ *has full column rank and that condition* (U5) *does not hold. Namely, the kernel of the* 441×252 *matrix* $\mathcal{C}_5(\mathbf{A}) \odot \mathcal{C}_5(\mathbf{B})$ *is spanned by a vector* $\widehat{\mathbf{d}}^5$ *associated with some* $\mathbf{d} \in \mathbb{F}^{10}$. *Let* $\mathbf{C}$ *be a* 7×10 *matrix such that* $\mathbf{d} \notin \operatorname{range}(\mathbf{C}^T)$. *Then* (W5) *holds for the triplet* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. *If additionally* $k_{\mathbf{C}} \geq 5$, *then* (1.12) *holds. Hence, by Proposition 1.22,* $r_{\mathcal{T}} = 10$ *and the CPD of* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_{10}$ *is unique.*

The same situation occurs for tensors with other dimensions (see Table 3.1).

4. Application to tensors with symmetric frontal slices and Indscal. In this section we consider tensors with symmetric frontal slices (SFS), which we will briefly call SFS-tensors. We are interested in PDs of which the rank-1 terms have the same symmetry. Such decompositions correspond to the INDSCAL model, as introduced by Carroll and Chang [1]. A similar approach may be followed in the case of full symmetry.

TABLE 3.1

Some cases where the rank and the uniqueness of the CPD of $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ may be easily obtained from Proposition 1.22 or its Corollary 1.23 (see Example 3.5). Matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are generated randomly. Simulations indicate that the dimensions of $\mathbf{A}$ and $\mathbf{B}$ cause the dimension of $\ker(\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B}))$ *to be equal to 1. Thus,* (Um) *and* (Wm) *may be easily checked.*

dimensions of $\mathcal{T}$, $I \times J \times K$	$r_{\mathcal{T}} = R$	m=R-K+2	dimensions of $\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B})$	(Um)	(Wm)
$4 \times 5 \times 6$	7	3	40×35	does not hold	holds
$4 \times 6 \times 14$	14	2	90×91	holds	holds
$5 \times 7 \times 7$	9	4	175×216	does not hold	holds
$6 \times 9 \times 8$	11	5	756×462	does not hold	holds
$7 \times 7 \times 7$	10	5	441×252	does not hold	holds

We start with definitions of SFS-rank, SFS-PD, and SFS-CPD.

DEFINITION 4.1. *A third-order SFS-tensor $\mathcal{T} \in \mathbb{F}^{I\times I\times K}$ is* SFS-rank-1 *if it equals the outer product of three nonzero vectors $\mathbf{a} \in \mathbb{F}^I$, $\mathbf{a} \in \mathbb{F}^I$ and $\mathbf{c} \in \mathbb{F}^K$.*

DEFINITION 4.2. *A* SFS-PD *of a third-order SFS-tensor $\mathcal{T} \in \mathbb{F}^{I\times I\times K}$ expresses $\mathcal{T}$ as a sum of SFS-rank-1 terms:*

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{a}_r \circ \mathbf{c}_r, \tag{4.1}$$

where $\mathbf{a}_r \in \mathbb{F}^I$, $\mathbf{c}_r \in \mathbb{F}^K$, $1 \le r \le R$.

DEFINITION 4.3. *The* SFS-rank *of a SFS-tensor $\mathcal{T} \in \mathbb{F}^{I\times I\times K}$ is defined as the minimum number of SFS-rank-1 tensors in a PD of $\mathcal{T}$ and is denoted by $r_{SFS,\mathcal{T}}$.*

DEFINITION 4.4. *A* SFS-CPD *of a third-order SFS-tensor $\mathcal{T}$ expresses $\mathcal{T}$ as a minimal sum of SFS-rank-1 terms.*

Note that $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ is a SFS-CPD of $\mathcal{T}$ if and only if $\mathcal{T}$ is an SFS-tensor, $\mathbf{A} = \mathbf{B}$, and $R = r_{SFS,\mathcal{T}}$.

Now we can define uniqueness of the SFS-CPD.

DEFINITION 4.5. *Let $\mathcal{T}$ be a SFS-tensor of SFS-rank R. The SFS-CPD of $\mathcal{T}$ is* unique *if $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R = [\bar{\mathbf{A}}, \bar{\mathbf{A}}, \bar{\mathbf{C}}]_R$ implies that there exist an $R \times R$ permutation matrix $\mathbf{\Pi}$ and $R \times R$ nonsingular diagonal matrices $\mathbf{\Lambda_A}$ and $\mathbf{\Lambda_C}$ such that*

$$\bar{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda_A}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda_C}, \quad \mathbf{\Lambda}_{\mathbf{A}}^2\mathbf{\Lambda_C} = \mathbf{I}_R.$$

EXAMPLE 4.6. *Some SFS-tensors admit both SFS-CPDs and CPDs of which the terms are not partially symmetric. For instance, consider the SFS-tensor $\mathcal{T} \in \mathbb{R}^{I\times I\times K}$ in which $\mathbf{I}_I$ is stacked K times. Let $\mathbf{E}$ denote the $K \times I$ matrix of which all entries are equal to one. Then $\mathcal{T} = [\mathbf{X}, (\mathbf{X}^{-1})^T, \mathbf{E}]_I$, is a CPD of $\mathcal{T}$ for any nonsingular $I \times I$ matrix $\mathbf{X}$. On the other hand, $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{E}]_I$, is a SFS-CPD of $\mathcal{T}$ for any orthogonal $I \times I$ matrix $\mathbf{A}$.*

The following result was obtained in [20]. We present the proof for completeness.

LEMMA 4.7. *Let $\mathcal{T}$ be a SFS-tensor of rank R and let the CPD of $\mathcal{T}$ be unique. Then $r_{SFS,\mathcal{T}} = r_{\mathcal{T}}$, and the SFS-CPD of $\mathcal{T}$ is also unique.*

Proof. Let $[\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ be a CPD of the SFS-tensor $\mathcal{T}$. Because of the symmetry we also have $\mathcal{T} = [\mathbf{B}, \mathbf{A}, \mathbf{C}]_R$. Since the CPD of $\mathcal{T}$ is unique, there exist an $R \times R$ permutation matrix $\mathbf{\Pi}$ and $R \times R$ nonsingular diagonal matrices $\mathbf{\Lambda_A}$, $\mathbf{\Lambda_B}$, and $\mathbf{\Lambda_C}$ such that $\mathbf{B} = \mathbf{A}\mathbf{\Pi}\mathbf{\Lambda_A}$, $\mathbf{A} = \mathbf{B}\mathbf{\Pi}\mathbf{\Lambda_B}$, $\mathbf{C} = \mathbf{C}\mathbf{\Pi}\mathbf{\Lambda_C}$ and $\Lambda_{\mathbf{A}}\Lambda_{\mathbf{B}}\mathbf{\Lambda_C} = \mathbf{I}_R$. Since the CPD is unique, by (1.18), we have $k_{\mathbf{C}} \ge 2$. Hence, $\mathbf{\Pi} = \mathbf{\Lambda_C} = \mathbf{I}_R$ and $\mathbf{B} = \mathbf{A}\mathbf{\Lambda_A}$.

Thus, any CPD of $\mathcal{T}$ is in fact a SFS-CPD. Hence, $r_{SFS,\mathcal{T}} = r_{\mathcal{T}}$, and the SFS-CPD of $\mathcal{T}$ is unique. □

REMARK 4.8. *To the authors' knowledge, it is still an open question whether there exist SFS-tensors with unique SFS-CPD but non-unique CPD.*

Lemma 4.7 implies that conditions guaranteeing uniqueness of SFS-CPD may be obtained from conditions guaranteeing uniqueness of CPD by just ignoring the SFS-structure. To illustrate this, we present SFS-variants of Corollaries 1.25 and 1.28.

PROPOSITION 4.9. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R$ and $m_{\mathbf{C}} := R - r_{\mathbf{C}} + 2$. Assume that*

(i) $k_{\mathbf{A}} + k_{\mathbf{C}} \geq R + 2$*;*

(ii) $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A})$ *has full column rank.*

Then $r_{SFS,\mathcal{T}} = R$ and the SFS-CPD of tensor $\mathcal{T}$ is unique.

Proof. From Corollary 1.25 it follows that $r_{\mathcal{T}} = R$ and that the CPD of tensor $\mathcal{T}$ is unique. The proof now follows from Lemma 4.7. □

REMARK 4.10. *Under the additional assumption $r_{\mathbf{C}} = R$, Proposition 4.9 was proved in [15].*

PROPOSITION 4.11. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R$ and $m_{\mathbf{A}} := R - r_{\mathbf{A}} + 2$. Assume that*

(i) $k_{\mathbf{A}} + \max(\min(k_{\mathbf{C}} - 1, k_{\mathbf{A}}), \min(k_{\mathbf{C}}, k_{\mathbf{A}} - 1)) \geq R + 1$*;*

(ii) $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C})$ *has full column rank.*

Then $r_{SFS,\mathcal{T}} = R$ and the SFS-CPD of tensor $\mathcal{T}$ is unique.

Proof. By Lemma 4.7 it is sufficient to show that $r_{\mathcal{T}} = R$ and that the CPD of tensor $\mathcal{T}$ is unique. Both these statements follow from Corollary 1.25 applied to the tensor $[\mathbf{A}, \mathbf{C}, \mathbf{A}]_R$. □

PROPOSITION 4.12. *Let $\mathcal{T} = [\mathbf{A}, \mathbf{A}, \mathbf{C}]_R$, $m_{\mathbf{A}} = R - r_{\mathbf{A}} + 2$ and $m_{\mathbf{C}} = R - r_{\mathbf{C}} + 2$. Assume that the matrices*

$$\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C}), \tag{4.2}$$

$$\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}) \tag{4.3}$$

have full column rank. Then $r_{SFS,\mathcal{T}} = R$ and the SFS-CPD of tensor $\mathcal{T}$ is unique.

Proof. From Corollary 1.28 it follows that $r_{\mathcal{T}} = R$ and that the CPD of tensor $\mathcal{T}$ is unique. The proof now follows from Lemma 4.7. □

5. Uniqueness beyond (Wm). In this section we discuss an example in which even condition (Wm) is not satisfied. Hence, CPD uniqueness does not follow from Proposition 1.13 or Proposition 1.14. A fortiori, it does not follow from Proposition 1.22, Corollaries 1.23–1.25, Proposition 1.26, and Corollaries 1.28–1.30. We show that uniqueness of the CPD can nevertheless be demonstrated by combining subresults.

In this section we will denote by $\omega(\mathbf{d})$ the number of nonzero components of $\mathbf{d}$ and we will write $\mathbf{a} \parallel \mathbf{b}$ if the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, that is there exists a nonzero number $c \in \mathbb{F}$ such that $\mathbf{a} = c\mathbf{b}$.

For easy reference we include the following lemma concerning second compound matrices.

LEMMA 5.1. *[7, Lemma 2.4 (1) and Lemma 2.5]*

(1) *Let the product $\mathbf{XYZ}$ be defined. Then the product $\mathcal{C}_2(\mathbf{X})\mathcal{C}_2(\mathbf{Y})\mathcal{C}_2(\mathbf{Z})$ is also defined and*

$$\mathcal{C}_2(\mathbf{XYZ}) = \mathcal{C}_2(\mathbf{X})\mathcal{C}_2(\mathbf{Y})\mathcal{C}_2(\mathbf{Z}).$$

(2) *Let $\mathbf{d} = \begin{bmatrix} d_1 & d_2 & \dots & d_R \end{bmatrix} \in \mathbb{F}^R$. Then $\mathcal{C}_2(\operatorname{Diag}(\mathbf{d})) = \operatorname{Diag}(\widehat{\mathbf{d}}^2)$.*
In particular, $\omega(\mathbf{d}) \leq 1$ if and only if $\widehat{\mathbf{d}}^2 = \mathbf{0}$ if and only if $\mathcal{C}_2(\operatorname{Diag}(\mathbf{d})) = \mathbf{0}$.

EXAMPLE 5.2. *Let* $\mathcal{T}_\alpha = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_5$, *where*

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \alpha \neq 0.$$

Then $r_{\mathbf{A}} = r_{\mathbf{B}} = r_{\mathbf{C}} = 4$, $k_{\mathbf{A}} = k_{\mathbf{B}} = k_{\mathbf{C}} = 2$, *and* $m := m_{\mathbf{A}} = m_{\mathbf{B}} = m_{\mathbf{C}} = 5-4+2 = 3$. *One can check that none of the triplets* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, $(\mathbf{B}, \mathbf{C}, \mathbf{A})$, $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ *satisfies condition* (Wm). *Hence, the rank and the uniqueness of the factor matrices of* $\mathcal{T}_\alpha$ *do not follow from Proposition 1.13 or Proposition 1.14. We prove that* $r_{\mathcal{T}_\alpha} = 5$ *and that the CPD* $\mathcal{T}_\alpha = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_5$ *is unique.*

(i) *A trivial verification shows that*

$$\mathbf{A} \odot \mathbf{B}, \quad \mathbf{B} \odot \mathbf{C}, \quad \mathbf{C} \odot \mathbf{A}, \qquad \textit{have full column rank,} \tag{5.1}$$

$$\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B}), \quad \mathcal{C}_2(\mathbf{B}) \odot \mathcal{C}_2(\mathbf{C}), \quad \mathcal{C}_2(\mathbf{C}) \odot \mathcal{C}_2(\mathbf{A}) \quad \textit{have full column rank.} \tag{5.2}$$

Elementary algebra yields

$$\omega(\mathbf{A}^T\mathbf{x}) = 1 \Leftrightarrow \ \mathbf{x} \| \ \mathbf{e}_1^4 \ \ \textit{or} \ \ \mathbf{x} \parallel \mathbf{e}_4^4, \tag{5.3}$$

$$\omega(\mathbf{B}^T\mathbf{y}) = 1 \Leftrightarrow \ \mathbf{y} \| \ \mathbf{e}_1^4 \ \ \textit{or} \ \ \mathbf{y} \parallel \mathbf{e}_3^4, \tag{5.4}$$

$$\omega(\mathbf{C}^T\mathbf{z}) = 1 \Leftrightarrow \ \mathbf{z} \| \ \mathbf{e}_2^4 \ \ \textit{or} \ \ \mathbf{z} \parallel \mathbf{e}_3^4. \tag{5.5}$$

(ii) *Consider a CPD* $\mathcal{T}_\alpha = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_{\bar{R}}$, *i.e.* $\bar{R} = r_{\mathcal{T}_\alpha}$ *is minimal. We have* $\bar{R} \leq 5$. *For later use we show that any three solutions of the equation* $\omega(\bar{\mathbf{C}}^T\mathbf{z}) = 1$ *are linearly dependent. Indeed, assume that there exist three vectors* $\mathbf{z}_1, \ \mathbf{z}_2, \ \mathbf{z}_3 \in \mathbb{F}^4$ *such that* $\omega(\bar{\mathbf{C}}^T\mathbf{z}_1) = \omega(\bar{\mathbf{C}}^T\mathbf{z}_2) = \omega(\bar{\mathbf{C}}^T\mathbf{z}_3) = 1$. *By* (1.2)–(1.3),

$$\mathbf{T}^{(1)} = (\bar{\mathbf{A}} \odot \bar{\mathbf{B}})\bar{\mathbf{C}}^T = (\mathbf{A} \odot \mathbf{B})\mathbf{C}^T, \tag{5.6}$$

$$\mathbf{T}^{(2)} = (\bar{\mathbf{B}} \odot \bar{\mathbf{C}})\bar{\mathbf{A}}^T = (\mathbf{B} \odot \mathbf{C})\mathbf{A}^T, \tag{5.7}$$

$$\mathbf{T}^{(3)} = (\bar{\mathbf{C}} \odot \bar{\mathbf{A}})\bar{\mathbf{B}}^T = (\mathbf{C} \odot \mathbf{A})\mathbf{B}^T. \tag{5.8}$$

From (5.6) *it follows that* $\mathbf{A}\mathrm{Diag}(\mathbf{C}^T\mathbf{z}_i)\mathbf{B}^T = \bar{\mathbf{A}}\mathrm{Diag}(\bar{\mathbf{C}}^T\mathbf{z}_i)\bar{\mathbf{B}}^T$, *and hence, by Lemma 5.1 (1),*

$$\mathcal{C}_2(\mathbf{A})\mathcal{C}_2(\mathrm{Diag}(\mathbf{C}^T\mathbf{z}_i))\mathcal{C}_2(\mathbf{B}^T) = \mathcal{C}_2(\bar{\mathbf{A}})\mathcal{C}_2(\mathrm{Diag}(\bar{\mathbf{C}}^T\mathbf{z}_i))\mathcal{C}_2(\bar{\mathbf{B}}^T) = \mathbf{O}, \qquad i \in \{1,2,3\},$$

which can also be expressed as

$$[\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})]\,\widehat{\mathbf{d}}_i^2 = \mathbf{0}, \qquad \mathbf{d}_i := \mathbf{C}^T\mathbf{z}_i, \qquad i \in \{1,2,3\}.$$

By (5.2), $\widehat{\mathbf{d}}_i^2 = \mathbf{0}$ *for* $i \in \{1,2,3\}$. *Since* $\mathbf{C}^T$ *has full column rank, Lemma 5.1 (2) implies that* $\omega(\mathbf{C}^T\mathbf{z}_1) = \omega(\mathbf{C}^T\mathbf{z}_2) = \omega(\mathbf{C}^T\mathbf{z}_3) = 1$. *From* (5.5) *it follows that at least two of the vectors* $\mathbf{z}_1$, $\mathbf{z}_2$, *and* $\mathbf{z}_3$ *are collinear. Hence, the vectors* $\mathbf{z}_1$, $\mathbf{z}_2$, *and* $\mathbf{z}_3$ *are linearly dependent.*

(iii) *Since* $\mathbf{A} \odot \mathbf{B}$ *and* $\mathbf{C}^T$ *have full column rank, from* (5.6) *and Sylvester's rank inequality it follows that*

$$r_{\bar{\mathbf{C}}^T} \geq r_{(\bar{\mathbf{A}} \odot \bar{\mathbf{B}})\bar{\mathbf{C}}^T} = r_{(\mathbf{A} \odot \mathbf{B})\mathbf{C}^T} \geq r_{\mathbf{A} \odot \mathbf{B}} + r_{\mathbf{C}^T} - 5 = 5 + 4 - 5 = 4.$$

In a similar fashion, from (5.7) *and* (5.8) *we obtain* $r_{\bar{\mathbf{A}}^T} \geq 4$ *and* $r_{\bar{\mathbf{B}}^T} \geq 4$, *respectively. We conclude that* $\bar{R} \geq r_{\bar{\mathbf{A}}} = r_{\bar{\mathbf{B}}} = r_{\bar{\mathbf{C}}} = 4$.

Let us show that $\bar{R} = 5$. To obtain a contradiction, assume that $\bar{R} = 4$. In this case, since $r_{\bar{\mathbf{C}}^T} = 4$, $\bar{\mathbf{C}}^T$ is a nonsingular square matrix. Then the columns of $\mathbf{Z} := (\bar{\mathbf{C}}^T)^{-1}$ are linearly independent solutions of the equation $\omega(\bar{\mathbf{C}}^T\mathbf{z}) = 1$, which is a contradiction with (ii). *Hence, $\bar{R} = 5$.*

(iv) *Let us show that $k_{\bar{\mathbf{C}}} \geq 2$. Conversely, assume that $k_{\bar{\mathbf{C}}} = 1$. Since $r_{\bar{\mathbf{C}}} = 4$, it follows that there exists exactly one pair of proportional columns of $\bar{\mathbf{C}}$. Without loss of generality we will assume that $\bar{\mathbf{C}}_4 \parallel \bar{\mathbf{C}}_5$. Hence, $\mathbb{F}^4 = \operatorname{range}(\bar{\mathbf{C}}) = \operatorname{span}\{\bar{\mathbf{C}}_1, \bar{\mathbf{C}}_2, \bar{\mathbf{C}}_3, \bar{\mathbf{C}}_4\}$. Let $\begin{bmatrix}\mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4\end{bmatrix} := (\begin{bmatrix}\bar{\mathbf{C}}_1 & \bar{\mathbf{C}}_2 & \bar{\mathbf{C}}_3 & \bar{\mathbf{C}}_4\end{bmatrix}^T)^{-1}$. Then $\omega(\bar{\mathbf{C}}^T\mathbf{z}_1) = \omega(\bar{\mathbf{C}}^T\mathbf{z}_2) = \omega(\bar{\mathbf{C}}^T\mathbf{z}_3) = 1$, which is a contradiction with* (ii).
In a similar fashion we can prove that $k_{\bar{\mathbf{A}}} \geq 2$ and $k_{\bar{\mathbf{B}}} \geq 2$. Thus, $\min(k_{\bar{\mathbf{A}}}, k_{\bar{\mathbf{B}}}, k_{\bar{\mathbf{C}}}) \geq 2$.

(v) *Assume that there exist indices i, j, k, l and nonzero values t_1, t_2, t_3, t_4 such that*

$$(\bar{\mathbf{A}}^T)_1 = t_1\mathbf{e}_i^5, \quad (\bar{\mathbf{A}}^T)_4 = t_2\mathbf{e}_j^5, \quad (\bar{\mathbf{B}}^T)_1 = t_3\mathbf{e}_k^5, \quad (\bar{\mathbf{B}}^T)_3 = t_4\mathbf{e}_l^5. \tag{5.9}$$

Here we show that (5.9) *implies the uniqueness of the CPD of $\mathcal{T}_\alpha$ and a fortiori the uniqueness of the third factor matrix. The latter implication will as such be instrumental in the proof of* (vi). *That assumption* (5.9) *really holds, and thus implies CPD uniqueness, will be demonstrated in* (vii).
Combination of (5.7), (5.8), *and* (5.9) *yields*

$$\begin{aligned} \alpha\mathbf{b}_2 \otimes \mathbf{c}_2 &= t_1\bar{\mathbf{b}}_i \otimes \bar{\mathbf{c}}_i, & \mathbf{c}_2 \otimes \mathbf{a}_2 &= t_3\bar{\mathbf{c}}_k \otimes \bar{\mathbf{a}}_k, \\ \mathbf{b}_5 \otimes \mathbf{c}_5 &= t_2\bar{\mathbf{b}}_j \otimes \bar{\mathbf{c}}_j, & \mathbf{c}_4 \otimes \mathbf{a}_4 &= t_4\bar{\mathbf{c}}_l \otimes \bar{\mathbf{a}}_l. \end{aligned}$$

We see that $\mathbf{b}_2 \parallel \bar{\mathbf{b}}_i$. Also, $\mathbf{c}_2 \parallel \bar{\mathbf{c}}_i$ and $\mathbf{c}_2 \parallel \bar{\mathbf{c}}_k$. Since $k_{\bar{\mathbf{C}}} \geq 2$, it follows that $i = k$. Therefore, also $\mathbf{a}_2 \parallel \bar{\mathbf{a}}_i$. It is now clear that $[\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2]_1 - [\bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{c}}_i]_1 = \beta[\mathbf{e}_1^4, \mathbf{e}_1^4, \mathbf{e}_1^4]$ for some $\beta \in \mathbb{F}$. Let

$$\mathcal{T}_\beta := \mathcal{T}_\alpha - [\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2]_1 + \beta[\mathbf{e}_1^4, \mathbf{e}_1^4, \mathbf{e}_1^4] = [\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}]_5 - [\bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{c}}_i]_1.$$

*Obviously, $\mathcal{T}_\beta$ is rank-*4. *We claim that $\beta = 0$. Indeed, if $\beta \neq 0$, then repeating steps* (i)–(iii) *for $\mathcal{T}_\alpha$ replaced by $\mathcal{T}_\beta$ we obtain that $\mathcal{T}_\beta$ is rank-*5, *which is a contradiction. Hence, $[\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2]_1 = [\bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{c}}_i]_1$.*
*What is left to show, is that the CPD of the rank-*4 *tensor $\mathcal{T}_\alpha - [\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2]_1$ is unique. Note that the matrix $\begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5\end{bmatrix}$ has full column rank. From* (5.2) *it follows that $\mathcal{C}_2(\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5\end{bmatrix}) \odot \mathcal{C}_2(\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5\end{bmatrix})$ also has full column rank. Hence, by Proposition 1.15, the CPD of $\mathcal{T}_\alpha - [\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2]_1$ is unique.*

(vi) *Let us show that $k_{\bar{\mathbf{A}}} = k_{\bar{\mathbf{B}}} = k_{\bar{\mathbf{C}}} = 2$. Conversely, assume that $k_{\bar{\mathbf{C}}} \geq 3$. Then $r_{\bar{\mathbf{B}}} + k_{\bar{\mathbf{C}}} \geq 4 + 3 \geq R + 2$. Recall from (iv) that $k_{\bar{\mathbf{B}}} \geq 2$. Hence, condition (K2) holds for $\bar{\mathbf{B}}, \bar{\mathbf{C}}$. By Proposition 1.12 (1),*

$$\mathcal{C}_2(\bar{\mathbf{B}}) \odot \mathcal{C}_2(\bar{\mathbf{C}}) \text{ has full column rank.} \tag{5.10}$$

Let $\mathbf{x} \in \mathbb{F}^4$. From (5.7) *it follows that*

$$\mathbf{B}\operatorname{Diag}(\mathbf{A}^T\mathbf{x})\mathbf{C}^T = \bar{\mathbf{B}}\operatorname{Diag}(\bar{\mathbf{A}}^T\mathbf{x})\bar{\mathbf{C}}^T,$$

Hence, by Lemma 5.1 (1),

$$\mathcal{C}_2(\mathbf{B})\mathcal{C}_2(\operatorname{Diag}(\mathbf{A}^T\mathbf{x}))\mathcal{C}_2(\mathbf{C}^T) = \mathcal{C}_2(\bar{\mathbf{B}})\mathcal{C}_2(\operatorname{Diag}(\bar{\mathbf{A}}^T\mathbf{x}))\mathcal{C}_2(\bar{\mathbf{C}}^T),$$

which can also be expressed as

$$[\mathcal{C}_2(\mathbf{B}) \odot \mathcal{C}_2(\mathbf{C})]\, \widehat{\mathbf{d}}^2_{\mathbf{A}} = \left[\mathcal{C}_2(\bar{\mathbf{B}}) \odot \mathcal{C}_2(\bar{\mathbf{C}})\right] \widehat{\mathbf{d}}^2_{\bar{\mathbf{A}}}, \tag{5.11}$$

where $\mathbf{d}_{\mathbf{A}} = \mathbf{A}^T\mathbf{x}$ *and* $\mathbf{d}_{\bar{\mathbf{A}}} = \bar{\mathbf{A}}^T\mathbf{x}$*. From* (5.2), (5.10)*, and Lemma 5.1 (2) it follows that*

$$\begin{aligned} \omega(\mathbf{A}^T\mathbf{x}) = 1 &\overset{\text{Lemma 5.1 (2)}}{\Longleftrightarrow} \widehat{\mathbf{d}}^2_{\mathbf{A}} = \mathbf{0} \overset{(5.2),(5.10),(5.11)}{\Longleftrightarrow} \widehat{\mathbf{d}}^2_{\bar{\mathbf{A}}} = \mathbf{0} \\ &\overset{\text{Lemma 5.1 (2)}}{\Longleftrightarrow} \omega(\bar{\mathbf{A}}^T\mathbf{x}) = 1. \end{aligned} \tag{5.12}$$

In a similar fashion we can prove that for $\mathbf{y} \in \mathbb{F}^4$,

$$\omega(\mathbf{B}^T\mathbf{y}) = 1 \Leftrightarrow \omega(\bar{\mathbf{B}}^T\mathbf{y}) = 1. \tag{5.13}$$

Therefore, by (i)*, there exist indices* i, j, k, l *and nonzero values* t_1, t_2, t_3, t_4 *such that* (5.9) *holds. It follows from step* (v) *that the matrices* $\mathbf{C}$ *and* $\bar{\mathbf{C}}$ *are the same up to permutation and column scaling. Hence,* $k_{\bar{\mathbf{C}}} = k_{\mathbf{C}} = 2$*, which is a contradiction with* $k_{\bar{\mathbf{C}}} \geq 3$*. We conclude that* $k_{\bar{\mathbf{C}}} < 3$*. On the other hand, we have from (iv) that* $k_{\bar{\mathbf{C}}} \geq 2$*. Hence,* $k_{\bar{\mathbf{C}}} = 2$*.*
In a similar fashion we can prove that $k_{\bar{\mathbf{A}}} = k_{\bar{\mathbf{B}}} = 2$*.*

(vii) *Since* $k_{\bar{\mathbf{A}}} = k_{\bar{\mathbf{B}}} = 2$*, both* $\bar{\mathbf{A}}$ *and* $\bar{\mathbf{B}}$ *have a rank-deficient* 4×3 *submatrix. Since* $r_{\bar{\mathbf{A}}} = r_{\bar{\mathbf{B}}} = 4$*, it follows that there exist vectors* $\mathbf{x}_1$*,* $\mathbf{x}_2$*,* $\mathbf{y}_1$*,* $\mathbf{y}_2$ *such that*

$$\omega(\bar{\mathbf{A}}^T\mathbf{x}_1) = \omega(\bar{\mathbf{A}}^T\mathbf{x}_2) = \omega(\bar{\mathbf{B}}^T\mathbf{y}_1) = \omega(\bar{\mathbf{B}}^T\mathbf{y}_2) = 1, \qquad \mathbf{x}_1 \not\parallel \mathbf{x}_2, \qquad \mathbf{y}_1 \not\parallel \mathbf{y}_2.$$

From (5.11)*–*(5.13) *it follows that* $\omega(\mathbf{A}^T\mathbf{x}_1) = \omega(\mathbf{A}^T\mathbf{x}_2) = \omega(\mathbf{B}^T\mathbf{y}_1) = \omega(\mathbf{B}^T\mathbf{y}_2) = 1$*. By* (5.3)*–*(5.4) *there exist indices* i, j, k, l *and nonzero values* t_1, t_2, t_3, t_4 *such that* (5.9) *holds. Hence, by* (v)*, the CPD of* $\mathcal{T}_\alpha$ *is unique.*

6. Generic uniqueness.

6.1. Generic uniqueness of unconstrained CPD. It was explained in [4,16] that the conditions $r_{\mathbf{C}} = R$ and (C2) in Proposition 1.15 hold generically when they hold for one particular choice of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. It was indicated that this implies that the CPD of an $I \times J \times K$ tensor $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ is generically unique whenever $K \geq R$ and $C_I^2 C_J^2 \geq C_R^2$. These conditions guarantee that the matrix $\mathbf{C}$ generically has full column rank and that the number of columns of the $C_I^2 C_J^2 \times C_R^2$ matrix $\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})$ does not exceed its number of rows. In this subsection we draw conclusions for the generic case from the more general Proposition 1.14 and Corollary 1.25.

As in [4,16], our proofs are based on the following lemma.

Lemma 6.1. *Let* $f(\mathbf{x})$ *be an analytic function of* $\mathbf{x} \in \mathbb{F}^n$ *and let* μ_n *be the Lebesgue measure on* $\mathbb{F}^n$*. If* $\mu_n\{\mathbf{x} :\ f(\mathbf{x}) = 0\} > 0$*, then* $f \equiv 0$*.*

Proof. The result easily follows from the uniqueness theorem for analytic functions (see for instance [11, Lemma 2, p. 1855]). ∎

The following corollary trivially follows from Lemma 6.1.

Corollary 6.2. *Let* $f(\mathbf{x})$ *be an analytic function of* $\mathbf{x} \in \mathbb{F}^n$ *and let* μ_n *be the Lebesgue measure on* $\mathbb{F}^n$*. Assume that there exists a point* $\mathbf{x}_0$ *such that* $f(\mathbf{x}_0) \neq 0$*. Then* $\mu_n\{\mathbf{x} :\ f(\mathbf{x}) = 0\} = 0$*.*

We will use the following matrix analogue of Corollary 6.2.

Lemma 6.3. *Let* $\mathbf{F}(\mathbf{x}) = (f_{pq}(\mathbf{x}))_{p,q=1}^{P,Q}$*, with* $P \geq Q$*, be an analytic matrix-valued function of* $\mathbf{x} \in \mathbb{F}^n$ *(that is, each entry* $f_{pq}(\mathbf{x})$ *is an analytic function of* $\mathbf{x}$*) and let* μ_n

be the Lebesgue measure on $\mathbb{F}^n$. Assume that there exists a point $\mathbf{x}_0$ such that $\mathbf{F}(\mathbf{x}_0)$ has full column rank. Then

$$\mu_n\{\mathbf{x}:\ \mathbf{F}(\mathbf{x}) \textit{ does not have full column rank}\} = 0.$$

Proof. Let $\mathbf{f}(\mathbf{x}) := \mathcal{C}_Q(\mathbf{F}(\mathbf{x}))$ and $L := C_P^Q$. Then $\mathbf{f}:\ \mathbb{F}^n \to \mathbb{F}^L : \mathbf{x} \to \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) & \dots & f_L(\mathbf{x})\end{bmatrix}^T$ is a vector-valued analytic function. Note that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ if and only if the matrix $\mathbf{F}(\mathbf{x})$ does not have full column rank. Since $\mathbf{f}(\mathbf{x}_0) \neq \mathbf{0}$, there exists $l_0 \in \{1,\dots,L\}$ such that $f_{l_0}(\mathbf{x}) \neq 0$. Hence, by Corollary 6.2, $\mu_n\{x:\ f_{l_0}(\mathbf{x}) = 0\} = 0$. Therefore,

$$\begin{aligned}&\mu_n\{\mathbf{x}:\ \mathbf{F}(\mathbf{x}) \text{ does not have full column rank}\} = \mu_n\{\mathbf{x}:\ \mathbf{f}(\mathbf{x}) = \mathbf{0}\}\\ =&\mu_n\left\{\bigcap_{l=1}^{L}\{x:\ f_l(\mathbf{x}) = 0\}\right\} \leq \mu_n\{x:\ f_{l_0}(\mathbf{x}) = 0\} = 0. \qquad \square\end{aligned}$$

The following lemma implies that, if $k_{\mathbf{C}} = r_{\mathbf{C}}$, then (1.12) in Proposition 1.20 holds generically, provided there exist matrices $\mathbf{A}_0 \in \mathbb{F}^{I\times R}$ and $\mathbf{B}_0 \in \mathbb{F}^{J\times R}$ for which $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ has full column rank.

LEMMA 6.4. *Suppose the matrices $\mathbf{A}_0 \in \mathbb{F}^{I\times R}$, $\mathbf{B}_0 \in \mathbb{F}^{J\times R}$, and $\mathbf{C} \in \mathbb{F}^{K\times R}$ satisfy the following conditions:*

$$k_{\mathbf{A}_0} = \min(I,R), \quad k_{\mathbf{B}_0} = \min(J,R), \quad k_{\mathbf{C}} = r_{\mathbf{C}}.$$

Suppose further the matrix $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ has full column rank, where $m = R - r_{\mathbf{C}} + 2$. Then

$$\max(\min(I, J-1, R-1), \min(I-1, J, R-1)) + k_{\mathbf{C}} \geq R + 1. \tag{6.1}$$

Proof. By Proposition 1.12 (2) and (9), $\min(k_{\mathbf{A}_0}, k_{\mathbf{B}_0}) \geq m_{\mathbf{C}}$. Hence,

$$\min(I,J,R) \geq \min(k_{\mathbf{A}_0}, k_{\mathbf{B}_0}) \geq m_{\mathbf{C}} = R - r_{\mathbf{C}} + 2 = R - k_{\mathbf{C}} + 2.$$

Therefore,

$$\begin{aligned}\max(\min(I, J-1, R-1), \min(I-1, J, R-1)) + k_{\mathbf{C}} &\geq \min(I-1, J-1) + k_{\mathbf{C}}\\ &\geq R - k_{\mathbf{C}} + 2 - 1 + k_{\mathbf{C}} = R + 1.\end{aligned}$$

Hence, (6.1) holds. $\square$

The following proposition is the main result of this section.

PROPOSITION 6.5. *Let the matrix $\mathbf{C} \in \mathbb{F}^{K\times R}$ be fixed and suppose $k_{\mathbf{C}} \geq 1$. Assume that there exist matrices $\mathbf{A}_0 \in \mathbb{F}^{I\times R}$ and $\mathbf{B}_0 \in \mathbb{F}^{J\times R}$ such that $\mathcal{C}_m(\mathbf{A}_0) \odot \mathcal{C}_m(\mathbf{B}_0)$ has full column rank, where $m = R - r_{\mathbf{C}} + 2$. Set $n = (I+J)R$. Then*

(i)

$$\begin{aligned}\mu_n\{(\mathbf{A},\mathbf{B}) : \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R &\textit{ has rank less than } R \textit{ or}\\ &\textit{the third factor matrix of } \mathcal{T} \textit{ is not unique}\} = 0.\end{aligned}$$

(ii) *If additionally, $k_{\mathbf{C}} = r_{\mathbf{C}}$, or (6.1) holds, then*

$$\begin{aligned}\mu_n\{(\mathbf{A},\mathbf{B}):\ \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R &\textit{ has rank less than } R \textit{ or}\\ &\textit{the CPD of } \mathcal{T} \textit{ is not unique}\} = 0.\end{aligned} \tag{6.2}$$

Proof. (i) Let $P := C_I^m C_J^m$, $Q := C_R^m$, $n := (I+J)R$, $\mathbf{x} := (\mathbf{A},\mathbf{B})$, $\mathbf{x}_0 := (\mathbf{A}_0,\mathbf{B}_0)$ and $\mathbf{F}(\mathbf{x}) := \mathcal{C}_m(\mathbf{A})\odot\mathcal{C}_m(\mathbf{B})$. Since $k_{\mathbf{C}}\geq 1$, from Proposition 1.14 and Lemma 6.3 it follows that

$$\begin{aligned}&\mu_n\{(\mathbf{A},\mathbf{B}) : \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R \text{ has rank less than } R \text{ or}\\ &\qquad\qquad \text{the third factor matrix of } \mathcal{T} \text{ is not unique}\}\\ &\leq \mu_n\{(\mathbf{A},\mathbf{B}) : \mathcal{C}_m(\mathbf{A})\odot\mathcal{C}_m(\mathbf{B}) \text{ does not have full column rank}\} = 0.\end{aligned}$$

(ii) By Lemma 6.4, we can assume that (6.1) holds. We obviously have

$$\mu_n\{(\mathbf{A},\mathbf{B}) :\ k_{\mathbf{A}} < \min(I,R) \ \text{ or } k_{\mathbf{B}} < \min(J,R)\} = 0.$$

Hence, by (6.1),

$$\mu_n\{(\mathbf{A},\mathbf{B}) :\ (1.12) \text{ does not hold}\} = 0.$$

From Proposition 1.20 and (i) it follows that

$$\begin{aligned}&\mu_n\{(\mathbf{A},\mathbf{B}) :\ \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R \text{ has rank less than } R \text{ or}\\ &\qquad\qquad \text{the CPD of } \mathcal{T} \text{ is not unique}\}\\ &\leq \mu_n\{(\mathbf{A},\mathbf{B}) :\ \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R \text{ has rank less than } R \text{ or}\\ &\qquad\qquad \text{the third factor matrix of } \mathcal{T} \text{ is not unique or}\\ &\qquad\qquad (1.12) \text{ does not hold}\} = 0. \qquad \square\end{aligned}$$

PROPOSITION 6.6. *The CPD of an $I\times J\times K$ tensor of rank R is generically unique if there exist matrices $\mathbf{A}_0\in\mathbb{F}^{I\times R}$ and $\mathbf{B}_0\in\mathbb{F}^{J\times R}$ such that $\mathcal{C}_m(\mathbf{A}_0)\odot\mathcal{C}_m(\mathbf{B}_0)$ has full column rank, where $m = R-\min(K,R)+2$.*

Proof. Generically we have $r_{\mathbf{C}} = \min(K,R)$. Let $N = (I+J+K)R$, $n = (I+J)R$, and let $\Omega = \{\mathbf{C} :\ k_{\mathbf{C}} < r_{\mathbf{C}}\}\subset\mathbb{F}^{KR}$. By application of Lemma 6.3, one obtains that $\mu_{KR}(\Omega) = 0$. From Proposition 6.5 it follows that (6.2) holds for $\mathbf{C}\notin\Omega$. Now

$$\begin{aligned}&\mu_N\{(\mathbf{A},\mathbf{B},\mathbf{C}) :\ \mathcal{T} := [\mathbf{A},\mathbf{B},\mathbf{C}]_R \text{ has rank less than } R \text{ or}\\ &\qquad\qquad \text{the CPD of } \mathcal{T} \text{ is not unique}\} = 0\end{aligned}$$

follows from Fubini's theorem [9, Theorem C, p. 148]. $\square$

Proof of Proposition 1.31. Proposition 1.31 follows from Proposition 6.6 by permuting factors. $\square$

6.2. Generic uniqueness of SFS-CPD. For generic uniqueness of the SFS-CPD we resort to the following definition.

DEFINITION 6.7. *Let μ be the Lebesgue measure on $\mathbb{F}^{(2I+K)R}$. The SFS-CPD of an $I\times I\times K$ tensor of SFS-rank R is generically unique if*

$$\mu\{(\mathbf{A},\mathbf{C}) :\ \textit{the SFS-CPD of the tensor } [\mathbf{A},\mathbf{A},\mathbf{C}]_R \textit{ is not unique }\} = 0.$$

We have the following counterpart of Proposition 1.31.

PROPOSITION 6.8. *The SFS-CPD of an $I\times I\times K$ SFS-tensor of SFS-rank R is generically unique if there exist matrices $\mathbf{A}_0\in\mathbb{F}^{I\times R}$ and $\mathbf{C}_0\in\mathbb{F}^{K\times R}$ such that $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0)\odot\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0)$ or $\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{A}_0)\odot\mathcal{C}_{m_{\mathbf{A}}}(\mathbf{C}_0)$ has full column rank, where $m_{\mathbf{C}} = R-\min(K,R)+2$ and $m_{\mathbf{A}} = R-\min(I,R)+2$.*

Proof. The proof is obtained by combining Proposition 1.31 and Lemma 4.7. $\square$

6.3. Examples. EXAMPLE 6.9. *This example illustrates how one may adapt the approach in subsections 6.1 and 6.2 to particular types of structured factor matrices.*

Let $\mathcal{I}_4$ *be the* $4 \times 4 \times 4$ *tensor with ones on the main diagonal and zero off-diagonal entries and let* $\mathcal{T} = \mathcal{I}_4 + \mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$ *be a generic rank-*1 *perturbation of* $\mathcal{I}_4$*. Then* $\mathcal{T} = [[\mathbf{I}_4\ \mathbf{a}], [\mathbf{I}_4\ \mathbf{b}], [\mathbf{I}_4\ \mathbf{c}]]_5$*. Since the k-ranks of all factor matrices of* $\mathcal{T}$ *are equal to* 4*, it follows from Kruskal's Theorem 1.8 that* $r_{\mathcal{T}} = 5$ *and that the CPD of* $\mathcal{T}$ *is unique.*

*Let us now consider structured rank-*1 *perturbations* $\bar{\mathbf{a}} \circ \bar{\mathbf{b}} \circ \bar{\mathbf{c}}$ *that do not change the fourth vertical, third horizontal, and second frontal slice of* $\mathcal{I}_4$*. The vectors* $\bar{\mathbf{a}}$*,* $\bar{\mathbf{b}}$*, and* $\bar{\mathbf{c}}$ *admit the following parameterizations*

$$\bar{\mathbf{a}} = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \end{bmatrix}, \quad \bar{\mathbf{b}} = \begin{bmatrix} b_1 & b_2 & 0 & b_4 \end{bmatrix}, \quad \bar{\mathbf{c}} = \begin{bmatrix} c_1 & 0 & c_3 & c_4 \end{bmatrix},$$

with $a_i, b_j, c_k \in \mathbb{F}$*.*

Now the k-ranks of all factor matrices of $\bar{\mathcal{T}} := \mathcal{I}_4 + \bar{\mathbf{a}} \circ \bar{\mathbf{b}} \circ \bar{\mathbf{c}}$ *are equal to* 3*, and (generic) uniqueness of the CPD of* $\bar{\mathcal{T}}$ *does not follow from Kruskal's Theorem 1.8.*

We show that

$$\mu_9\{(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) : \textit{the CPD of } \bar{\mathcal{T}} := \mathcal{I}_4 + \bar{\mathbf{a}} \circ \bar{\mathbf{b}} \circ \bar{\mathbf{c}} \textit{ is not unique or } r_{\bar{\mathcal{T}}} < 5\} = 0,$$

*that is, the CPD of rank-*1 *structured generic perturbation of* $\mathcal{I}_4$ *is again unique.*

Let the matrices $\mathbf{A}_0$*,* $\mathbf{B}_0$*, and* $\mathbf{C}_0$ *be given by the matrices* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{C}$*, respectively, in Example 3.2. As in Example 3.2 the matrix pairs* $(\mathbf{A}_0, \mathbf{B}_0)$*,* $(\mathbf{B}_0, \mathbf{C}_0)$*, and* $(\mathbf{C}_0, \mathbf{A}_0)$ *satisfy condition* (C3)*. Then, by Lemma 6.3,*

$$\mu_6\{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) : \mathcal{C}_3(\mathbf{A}) \odot \mathcal{C}_3(\mathbf{B}) \textit{ does not have full column rank}\} = 0,$$
$$\mu_6\{(\bar{\mathbf{b}}, \bar{\mathbf{c}}) : \mathcal{C}_3(\mathbf{B}) \odot \mathcal{C}_3(\mathbf{C}) \textit{ does not have full column rank}\} = 0.$$

By Fubini's theorem [9, Theorem C, p. 148],

$$\mu_9\{(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) : \mathcal{C}_3(\mathbf{A}) \odot \mathcal{C}_3(\mathbf{B}) \textit{ has not full column rank or } k_{\mathbf{C}} < 3 \textit{ or } r_{\mathbf{C}} < 4\} = 0,$$
$$\mu_9\{(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) : \mathcal{C}_3(\mathbf{B}) \odot \mathcal{C}_3(\mathbf{C}) \textit{ has not full column rank or } k_{\mathbf{A}} < 3 \textit{ or } r_{\mathbf{A}} < 4\} = 0.$$

*Now generic uniqueness of the structured rank-*1 *perturbation of* $\mathcal{I}_4$ *follows from Proposition 1.21.*

EXAMPLE 6.10. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *denote a PD of an* $I \times I \times (2I-1)$ *tensor, where* $I \geq 4$*. Generically,* $k_{\mathbf{A}} = k_{\mathbf{B}} = I$ *and* $k_{\mathbf{C}} = 2I-1$*. Then Kruskal's condition* (1.4) *guarantees generic uniqueness for* $R \leq \lfloor \frac{I+I+2I-1-2}{2} \rfloor = 2I-1$*.*

On the other hand, (1.11) guarantees generic uniqueness of the CPD under the conditions $R \leq 2I-1$ *and* $C_R^2 \leq (C_I^2)^2$*. The maximum value of R that satisfies these bounds is shown in the column corresponding to* $m_{\mathbf{C}} = 2$ *in Table 6.1. The condition in Theorem 1.18 is even more relaxed.*

We now move to cases where $R > 2I-1$*, where Theorem 1.18 no longer applies. By Proposition 6.6, the CPD of* $\mathcal{T}$ *of an* $I \times I \times (2I-1)$ *tensor of rank R is generically unique if there exist matrices* $\mathbf{A}_0 \in \mathbb{F}^{I \times R}$ *and* $\mathbf{B}_0 \in \mathbb{F}^{I \times R}$ *such that* $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank, where* $m_{\mathbf{C}} = R - (2I-1) + 2 = R - 2I + 3$*. The proof of Proposition 6.6 shows that, if there exist* $\mathbf{A}_0$ *and* $\mathbf{B}_0$ *such that* $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank, then actually* $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank with probability one when* $\mathbf{A}_0$ *and* $\mathbf{B}_0$ *are drawn from continuous distributions. Hence, we generate random* $\mathbf{A}_0$ *and* $\mathbf{B}_0$ *and check up to which value of R the matrix* $\mathcal{C}_{m_{\mathbf{C}}}(\mathbf{A}_0) \odot \mathcal{C}_{m_{\mathbf{C}}}(\mathbf{B}_0)$ *has full column rank. Table 6.1 shows the results for* $4 \leq I \leq 9$*.*

For instance, we obtain that the CPD of a $9 \times 9 \times 17$ *tensor of rank* R *is generically unique if* $R \leq 20$. *(By of comparison, Theorem 1.18 only guarantees uniqueness up to* $R = 17$.*)*

Proposition 6.6 corresponds to condition (i) in Proposition 1.31. Note that, for $R \geq 2I - 1$, *we generically have* $m_{\mathbf{A}} = m_{\mathbf{B}} = R - I + 2 \geq I + 1$ *such that the* $m_{\mathbf{B}}$*-th compound matrix of* $\mathbf{A}$ *and the* $m_{\mathbf{A}}$*-th compound matrix of* $\mathbf{B}$ *are not defined. Hence, we cannot resort to condition (ii) or (iii) in Proposition 1.31.*

Table 6.1

Upper bounds on R *under which generic uniqueness of the CPD of an* $I \times I \times (2I-1)$ *tensor is guaranteed by Proposition 6.6.*

dimensions of $\mathcal{T}$	$m = R - 2I + 3$			
$I \times I \times (2I-1)$	2	3	4	5
$4 \times 4 \times 7$	7			
$5 \times 5 \times 9$	9			
$6 \times 6 \times 11$	11	12		
$7 \times 7 \times 13$	13	14		
$8 \times 8 \times 15$	15	16	17	
$9 \times 9 \times 17$	17	18	19	20

Remark 6.11. *For* $I = 3$ *and* $R = 2I - 1 = 5$, *the CPD of an* $I \times I \times (2I-1)$ *tensor* $\mathcal{T}$ *is not generically unique [19], [17]. This is the reason why in Table 6.1 we start from* $I = 4$.

Remark 6.12. *It was shown in [11, Corollary 1, p.1852] that the matrix* $\mathcal{C}_1(\mathbf{A}) \odot \mathcal{C}_1(\mathbf{B}) = \mathbf{A} \odot \mathbf{B}$ *has full column rank with probability one when the number of rows of* $\mathbf{A} \odot \mathbf{B}$ *does not exceed its number of columns. The same statement was made for the matrix* $\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})$ *in [4, 16]. However, the statement does not hold for compound matrices of arbitrary order. For instance, it does not hold for* $\mathcal{C}_5(\mathbf{A}) \odot \mathcal{C}_5(\mathbf{B})$, *where* $\mathbf{A} \in \mathbb{F}^{6\times 9}$ *and* $\mathbf{B} \in \mathbb{F}^{7\times 9}$.

Example 6.13. *Let* $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_9$ *denote a generic PD in 9 terms in the* $6 \times 7 \times 6$ *case. Then* $m_{\mathbf{A}} = m_{\mathbf{C}} = 9 - 6 + 2 = 5$ *and* $m_{\mathbf{B}} = 9 - 7 + 2 = 4$. *The matrices* $\mathbf{M_C} := \mathcal{C}_5(\mathbf{A}) \odot \mathcal{C}_5(\mathbf{B})$ *and* $\mathbf{M_A} := \mathcal{C}_5(\mathbf{B}) \odot \mathcal{C}_5(\mathbf{C})$ *have* $C_6^5 C_7^5 = C_9^5 = 126$ *rows and columns. Numerical experiments indicate that* $\dim\ker(\mathbf{M_C}) = \dim\ker(\mathbf{M_A}) = 15$ *with probability one. Hence, we cannot use Proposition 1.31* (i) *or* (ii) *for proving uniqueness of the CPD. On the other hand, the* $C_6^4 C_6^4 \times C_9^4$ *(*225×126*) matrix* $\mathbf{M_B} := \mathcal{C}_4(\mathbf{C}) \odot \mathcal{C}_4(\mathbf{A})$ *turns out to have full column rank for a random choice of* $\mathbf{A}$ *and* $\mathbf{C}$. *Hence, by Proposition 1.31* (iii), *the CPD is generically unique.*

Example 6.14. *Here we consider* $I \times I \times K$ *tensors with* $I \in \{4, \ldots, 9\}$ *and* $K \in \{2, \ldots, 33\}$, *which is more general than Example 6.10.*

We check up to which value of R *one of the conditions in Proposition 1.31 holds for a random choice of the factor matrices. Up to this value the CPD is generically unique. The results are shown as the left-most values in Table 6.2. We also check up to which value of* R *one of the conditions in Proposition 6.8 holds for a random choice of the factor matrices. Up to this value the SFS-CPD is generically unique. The results are shown as the middle values in Table 6.2.*

The right-most values correspond to the maximum value of R *for which generic uniqueness is guaranteed by Kruskal's Theorems 1.8–1.10, i.e., the largest value of* R *that satisfies* $2\min(I,R) + \min(K,R) \geq 2R + 2$. *Note that Kruskal's bound is the same for CPD and SFS-CPD. The bold values in the table correspond to the results that were not yet covered by Kruskal's Theorems 1.8–1.10 or Proposition 1.15 (*$m = 2$*).*

Table 6.2

Upper bounds on R under which generic uniqueness of the CPD (left and right value) and SFS-CPD (middle and right value) of an $I\times I\times K$ tensor is guaranteed by Proposition 1.31 (left), Proposition 6.8 (middle), and Kruskal's Theorems 1.8–1.10 (right). The values shown in bold correspond to the results that were not yet covered by Kruskal's Theorems 1.8–1.10 or Proposition 1.15 ($m=2$).

		I					
		4	5	6	7	8	9
K	2	4, 4, 4	5, 5, 5	6, 6, 6	7, 7, 7	8, 8, 8	9, 9, 9
	3	4, 4, 4	5, 5, 5	6, 6, 6	7, 7, 7	8, 8, 8	9, 9, 9
	4	5, 5, 5	6, 6, 6	7, 7, 7	8, 8, 8	9, 9, 9	10, 10, 10
	5	5, 5, 5	6, 6, 6	7, 7, 7	8, 8, 8	**10**, **10**, 9	**11**, **11**, 10
	6	6, 6, 6	7, 7, 7	8, 8, 8	9, 9, 9	10, 10, 10	11, 11, 11
	7	7, 6, 6	**8**, 7, 7	**9**, 8, 8	9, 9, 9	**11**, **11**, 10	**12**, **12**, 11
	8	8, 6, 6	**9**, 8, 8	9, 9, 9	10, 10, 10	11, 11, 11	12, 12, 12
	9	9, 6, 6	9, 9, 8	**10**, **10**, 9	**11**, 10, 10	**12**, 11, 11	**13**, **13**, 12
	10	9, 6, 6	10, 10, 8	**11**, 10, 10	**12**, 11, 11	**13**, 12, 12	**14**, 13, 13
	11	9, 6, 6	11, 10, 8	**12**, 11, 10	**13**, **12**, 11	**14**, **13**, 12	**15**, **14**, 13
	12	9, 6, 6	12, 10, 8	**13**, 12, 10	**14**, **13**, 12	**15**, **14**, 13	**15**, **15**, 14
	13	9, 6, 6	13, 10, 8	**14**, 13, 10	**14**, **14**, 12	**15**, **15**, 13	**16**, **15**, 14
	14	9, 6, 6	14, 10, 8	14, 14, 10	**15**, **15**, 12	**16**, **15**, 14	**17**, **16**, 15
	15	9, 6, 6	14, 10, 8	15, 15, 10	**16**, 15, 12	**17**, **16**, 14	**18**, **17**, 15
	16	9, 6, 6	14, 10, 8	16, 15, 10	**17**, 16, 12	**18**, **17**, 14	**19**, **18**, 16
	17	9, 6, 6	14, 10, 8	17, 15, 10	**18**, 17, 12	**19**, **18**, 14	**20**, **19**, 16
	18	9, 6, 6	14, 10, 8	18, 15, 10	**19**, 18, 12	**20**, **19**, 14	**20**, **20**, 16
	19	9, 6, 6	14, 10, 8	19, 15, 10	**20**, 19, 12	**20**, **20**, 14	**21**, **20**, 16
	20	9, 6, 6	14, 10, 8	20, 15, 10	20, 20, 12	**21**, 20, 14	**22**, **21**, 16
	21	9, 6, 6	14, 10, 8	21, 15, 10	21, 20, 12	**22**, 21, 14	**23**, **22**, 16
	22	9, 6, 6	14, 10, 8	21, 15, 10	22, 20, 12	**23**, 22, 14	**24**, **23**, 16
	23	9, 6, 6	14, 10, 8	21, 15, 10	23, 20, 12	**24**, 23, 14	**25**, **24**, 16
	24	9, 6, 6	14, 10, 8	21, 15, 10	24, 20, 12	**25**, 24, 14	**26**, **25**, 16
	25	9, 6, 6	14, 10, 8	21, 15, 10	25, 20, 12	**26**, 25, 14	**26**, 25, 16
	26	9, 6, 6	14, 10, 8	21, 15, 10	26, 20, 12	**27**, 26, 14	**27**, 26, 16
	27	9, 6, 6	14, 10, 8	21, 15, 10	27, 20, 12	27, 26, 14	**28**, 27, 16
	28	9, 6, 6	14, 10, 8	21, 15, 10	28, 20, 12	28, 26, 14	**29**, 28, 16
	29	9, 6, 6	14, 10, 8	21, 15, 10	29, 20, 12	29, 26, 14	**30**, 29, 16
	30	9, 6, 6	14, 10, 8	21, 15, 10	30, 20, 12	30, 26, 14	**31**, 30, 16
	31	9, 6, 6	14, 10, 8	21, 15, 10	30, 20, 12	31, 26, 14	**32**, 31, 16
	32	9, 6, 6	14, 10, 8	21, 15, 10	30, 20, 12	32, 26, 14	**33**, 32, 16
	33	9, 6, 6	14, 10, 8	21, 15, 10	30, 20, 12	33, 26, 14	**34**, 33, 16

Remark 6.15. *Most of the improved left-most values in Table 6.2 also follow from Theorems 1.16, 1.18–1.19. (Concerning the latter, if the CPD of an $I\times I\times I$ tensor of rank R is generically unique for $R\leq k(I)$, then a forteriori the CPD of a rank-R $I\times I\times K$ tensor with $K>I$ is generically unique for $R\leq k(I)$.) An important difference is that our bounds remain valid for many constrained CPDs. We briefly give two examples. Rather than going into details, let us suffice by mentioning that (generic) uniqueness in these examples may be defined and studied in the same way as it was done in Subsections 6.1 and 6.2 for unsymmetric CPD and SFS-CPD, respectively.*

1. *Let the third factor matrix of $I\times I\times K$ tensor $\mathcal{T}$ belong to a class of structured matrices Ω such that the condition $I+k_{\mathbf{C}}\geq R+2$ is valid for generic $\mathbf{C}\in\Omega$. An example of a class for which this may be true, is the class of $K\times R$ Hankel matrices. In Subsection 6.1 Proposition 6.5 leads to Proposition 1.31 for unconstrained CPD. Similarly, Proposition 6.5 with condition* (6.1) *replaced by condition $I+k_{\mathbf{C}}\geq R+2$ leads to an analogue of Proposition*

1.31 that guarantees that a CPD with the third factor matrix belonging to Ω is generically unique for R bounded by the values in Table 6.2 (left values for unconstrained first and second factor matrices, and middle values in the case of partial symmetry).

2. *Let us now assume that the third factor matrix is unstructured and that the first two matrices have Toeplitz structure. Random Toeplitz matrices also yield the values in Table 6.2. Hence, such a constrained CPD is again generically unique for R bounded by the values in Table 6.2.*

REMARK 6.16. *In the case $r_{\mathbf{C}} = R$, both* (C2) *and* (U2) *are sufficient for overall CPD uniqueness, see* (1.9)*. In the case of* (C2)*, we generically have condition* (1.11)*. The more relaxed generic condition derived from* (U2) *is given in Theorem 1.18. For the case $r_{\mathbf{C}} < R$ we have obtained the deterministic result in Corollary 1.25 and its its generic version Proposition 1.31, both based on condition* (Cm)*. This suggests that by starting from Corollary 1.23, based on* (Um)*, more relaxed generic uniqueness results may be obtained.*

On the other hand, in Example 3.5 we have studied CPD of a rank-10 ($7 \times 7 \times 7$) tensor. Simulations along the lines of Example 3.5 suggest that condition (W5) *holds for random factor matrices, which then implies generic overall CPD uniqueness for $R = 10$. Starting from* (C5) *we have only demonstrated generic uniqueness up to $R = 9$, see the entry for $I = K = 7$ in Table 6.2. This suggests that by starting from Proposition 1.22, based on* (Wm)*, further relaxed generic uniqueness results may be obtained.*

7. Conclusion. Using results obtained in Part I [7], we have obtained new conditions guaranteeing uniqueness of a CPD. In the framework of the new uniqueness theorems, Kruskal's theorem and the existing uniqueness theorems for the case $R = r_{\mathbf{C}}$ are special cases. We have derived both deterministic and generic conditions.

The results can be easily adapted to the case of PDs in which one or several factor matrices are equal, such as INDSCAL. In the deterministic conditions the equalities can simply be substituted. In the generic setting one checks the same rank constraints as in the unconstrained case for a random example. The difference is that there are fewer independent entries to draw randomly. This may decrease the value of R up to which uniqueness is guaranteed. However, the procedure for determining this maximal value is completely analogous. The same holds true for PDs in which one or several factor matrices have structure (Toeplitz, Hankel, Vandermonde, etc.).

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